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NONLINEAR FILTERING AND APPROXIMATION TECHNIQUES

Final Report — September 1991

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Nonlinear Filtering and Approximation Techniques

This report presents the results obtained by the contractors in the study of partially observed diffusions. Because nonlinear filtering is a central theme in the works reported below, we begin with a brief presentation of this problem.

Let $\{(X_t, Y_t), t \geq 0\}$ be a pair of stochastic processes satisfying

$$\begin{aligned} dX_t &= b(X_t)\,dt + \sigma(X_t)\,dW_t + \rho(X_t)\,dV_t \\ dY_t &= h(X_t)\,dt + dV_t \end{aligned} \tag{1}$$

where $\{(W_t, V_t), t \geq 0\}$ are two independent Wiener processes, and the initial state X_0 is a random variable independent of $\{(W_t, V_t), t \geq 0\}$. The process $\{X_t, t \geq 0\}$ is not observed. We observe $\{Y_t, t \geq 0\}$ and we seek to estimate the current state X_t given the information available at time t, i.e. given $\mathcal{Y}_t = \sigma(Y_s, 0 \leq s \leq t)$.

Note that the choice of the above model (1) means that the state process $\{X_t, t \geq 0\}$ is a continuous Markov process.

The best estimate in the mean square error sense, of any function of the unknown r.v. X_t say $\phi(X_t)$, based on $\mathcal{Y}_t$, is the conditional mean $\mathbf{E}[\phi(X_t) \mid \mathcal{Y}_t]$, and computing this quantity for any function $\phi(\cdot)$ reduces to computing the conditional law of X_t given $\mathcal{Y}_t$. Assuming that this conditional law has a density with respect to the Lebesgue measure, it is well known that an unnormalized version $p_t(x)$ of the conditional density satisfies a recursive equation, actually a stochastic PDE called the Zakai equation

$$dp_t = L^* p_t\,dt + \sum_{k=1}^{d} B_k^* p_t\,dY_t^k \tag{2}$$

where L^* and B_k^* are the adjoint in the L^2 sense of the partial differential operators

$$L = \frac{1}{2} \sum_{i,j=1}^{m} a^{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b^i \frac{\partial}{\partial x_i} \qquad \text{and} \qquad B_k = h_k + \sum_{i=1}^{m} \rho_k^i \frac{\partial}{\partial x_i}$$

respectively. Note that at each time t, $p_t(\cdot)$ is a random function of the state variable x, i.e. a random element of an infinite dimensional space. This is of course a serious problem for practical implementation.

Let us describe the results which have been obtained during the period covered by the present contract.

1 ZPB

The purpose of this software is to make available the experience accumulated by the contractors about numerical techniques in nonlinear filtering. The basic idea is, given a formal description of both the model (1) and the problem to be solved (e.g. filtering, smoothing, hypotheses testing, etc.), to produce a program for the numerical solution of the corresponding Zakai equation (2).

The current version of ZPB is based on the computer algebra system Maple[1]. The main improvement over previous versions, is the existence of a user interface and graphical tools based on the X Window system[2]. The interface guides the user through the following steps

- definition/modification of the model and the problem to be solved, and automatic generation of the corresponding Fortran program,
- definition/modification of numerical values related with either the model or the algorithm, execution of the Fortran program, and visualization of the results,
- saving/loading of interesting examples.

Graphical tools are available to visualize the results of the computation

- a representation of both the simulated state trajectory and the estimated state trajectory vs. time is provided in a first window, as well as a shaded area representing some confidence region for the conditional distribution at a given level,
- a time can be selected in the first window, and a representation of the conditional density at the selected time is then provided in a second window,
- it is also possible to visualize in the second window, the continuous time evolution of the conditional density.

A short document presenting what is currently available in ZPB, and what is to be developed in the near future, is joint to the report.

2 Discretization of the Zakai equation

We are interested in studying numerical methods for the approximate solution of the Zakai equation (2)

$$dp_t = L^* p_t \, dt + \sum_{k=1}^{d} B_k^* p_t \, dY_t^k$$

where L is a second order partial differential operator, and B_k are first order partial differential operators.

[1]Maple is a registered trademark of Waterloo Maple Software.
[2]The X Window system is a trademark of the MIT.

In the case of independent noise, where $\rho \equiv 0$ in equation (1) so that $B_k = h_k$ are "zero-th order" partial differential operators, we have studied in [10] time discretization schemes based on splitting-up approximation, with error estimate of order $O(\Delta)$ and $O(\Delta^{3/2})$ where Δ is the time step. A probabilistic interpretation of the discrerization schemes was also provided.

In the case of correlated noise, we have proposed in [3] a time discretisation scheme based on the same splitting-up approach, with error estimate of order $O(\Delta)$. The correction part in the splitting-up approximation. is related with a degenerate stochastic PDE. for which a representation result in terms of stochastic characteristics can be found in [9] and [8]. We have obtained a time discretization scheme for the degenerate stochastic PDE based on Euler approximation of the stochastic characteristics, with error estimate of order $O(\Delta^{1/2})$.

Finally, extending the results of Raviart in the deterministic case [20], we have proposed in [4] a space discretization based on particle approximation, for first order stochastic PDE in Stratonovich form, which are degenerate second order stochastic PDE.

In relation with the design of finite time observers for deterministic partially observed systems presented in [5], full discretization schemes have been introduced in the special case of noise free state equations, where both $\sigma \equiv 0$ and $\rho \equiv 0$ in equation (1), see [7].

3 Filtering of piecewise linear systems

□ Continuous time systems

We consider a multi-dimensional stochastic system with dim X = dim $Y = m$. described by

$$\begin{aligned} dX_t &= b(X_t)\,dt + f(X_t)\,dV_t + g(X_t)\,dW_t \\ dY_t &= h(X_t)\,dt + \varepsilon\,dW_t \end{aligned} \tag{3}$$

where ε is a small parameter.

Our interest is for the situation where the coefficients are linear (or constant) on each component of a finite polyhedral partition $\{\Theta_i,\ 1 \le i \le l\}$ of the state space $\mathbf{R}^m$. For the sake of simplicity we assume that $l = 2$. i.e. $\Theta_- = \{x : \langle x, u\rangle < 0\}$ and $\Theta_+ = \{x : \langle x, u\rangle \ge 0\}$ where u is some non zero vector of $\mathbf{R}^m$. Let $\Delta = \{x : \langle x, u\rangle = 0\}$ denote the separating hyperplane. We assume that the coefficients of (3) satisfy

$$\begin{array}{rcc} & x \in \Theta_- & x \in \Theta_+ \\ b(x) = & B_- x & B_+ x \\ f(x) = & F_- & F_+ \\ g(x) = & G_- & G_+ \\ h(x) = & H_- x & H_- x \end{array}$$

and we assume that both H_- and H_+ are invertible. The case where $h(\cdot)$ is one-to-one has been considered in the previous contract, and we asume here that $h(\cdot)$ is not globally injective.

On each of the half spaces Θ_- and Θ_+ we have a linear system with non Gaussian initial condition. If we knew that the state would remain in a given half space for a certain time interval, then it would be natural to approximate the optimal nonlinear filter by the Kalman-Bucy filter associated with that half space. The design of an approximate filter is based on this idea.

- two Kalman filters $\widehat{X}_t^+$ and $\widehat{X}_t^-$ are considered which are associated with the two linear systems corresponding to the original piecewise linear system,
- a first test is used to find a time interval $[a,b]$ such that X_t does not cross the separating hyperplane Δ in $[a,b]$,
- provided that such a time interval $[a,b]$ has been found, a second test is used to decide whether $X_t \in \Theta_-$ or $X_t \in \Theta_+$ on $[a,b]$, i.e. to decide which Kalman filter $\widehat{X}_t^+$ or $\widehat{X}_t^-$ to follow on $[a,b]$.

We have proved in [19] that an hyperplane-crossing test can be designed with exponentially small probability of error, and that it is possible to design a test to decide between Θ_- and Θ_+, under either one of the following *detectability hypothesis*

$$(\mathrm{DH}_1) \qquad H_- \Sigma_- H_-^* \neq H_+ \Sigma_+ H_+^*$$

$$(\mathrm{DH}_2) \qquad \begin{cases} \Gamma = H_- \Sigma_- H_-^* = H_+ \Sigma_+ H_+^* \\ \ker\{H_- B_- H_-^{-1} - H_+ B_+ H_+^{-1}\} \subset \Delta \\ \text{the matrix } \Gamma^{-1}\{H_- B_- H_-^{-1} - H_+ B_+ H_+^{-1}\} \text{ is symmetric} \end{cases}$$

where $\Sigma_- = F_- F_-^* + G_- G_-^*$ and similarily for Σ_+.

The main difference between the two *detectability hypothesis* is that under (DH_1), we can decide almost instantaneously with an exponentially small probability of error, whereas under (DH_2), we need the interval $[a,b]$ to be long enough (actually almost infinite) in order to get an exponentially small probability of error.

Some examples in the case where $\dim X > \dim Y$ have been also considered in [14], [18], and [19].

□ Discrete time systems

We consider a one-dimensional discrete time stochastic dynamical system described by

$$x_{k+1} = x_k + \varepsilon\, b(x_k) + \sqrt{\varepsilon}\, \sigma(x_k)\, w_k$$

$$y_k = h(x_k) + \sqrt{\varepsilon}\, v_k$$

where ε is a small parameter. Such a system results e.g. from the discretization, with time step $\Delta t = \varepsilon$, of a continuous time system with small observation noise, such as (3) above.

Our interest is for the situation where the coefficients are piecewise linear (or constant) i.e.

$$
\begin{array}{rcc}
 & x < 0 & x \geq 0 \\
b(x) = & B_- x & B_+ x \\
\sigma(x) = & \sigma_- & \sigma_+ \\
h(x) = & H_- x & H_- x
\end{array}
$$

In the case where $h(\cdot)$ is not one–to–one (i.e. $H_+ \, H_- < 0$), we introduce the following *detectability hypotheses*

$$
(\mathrm{DH}_1) \qquad H_-^2 \, \sigma_-^2 \neq H_+^2 \, \sigma_+^2
$$

$$
(\mathrm{DH}_2) \qquad \left\{ \begin{array}{c} H_-^2 \, \sigma_-^2 = H_+^2 \, \sigma_+^2 \\ B_- \neq B_+ \end{array} \right.
$$

The case where hypothesis (DH_1) holds has been considered in [2]. Under hypothesis (DH_2), we have proved in [15] that an efficient approximate filter can be built, which is based on the same following idea than in the continuous time situation

- ◇ two Kalman filters $\widehat{x}_k^+$ and $\widehat{x}_k^-$ are considered which are associated with the two linear systems corresponding to the original piecewise linear system,
- ◇ a first test is used to find a time interval $[a, b]$ such that x_k does not cross the zero axis in $[a, b]$ with high probability.
- ◇ provided that such a time interval $[a, b]$ has been found, a second test is used to decide on the sign of x_k in $[a, b]$, i.e. to decide which Kalman filter $\widehat{x}_k^+$ or $\widehat{x}_k^-$ to follow.

Using the same heuristic approach as in [2], i.e. approximating some discrete processes by diffusion processes, explicit expressions have been obtained for the selection of thresholds.

Some numerical experiments have been performed on various examples, and the proposed approximate filter has been compared with the optimal filter obtained from the numerical solution of the corresponding Zakai equation. It is worth to mention that the Fortran programs for the numerical solution of the Zakai equations, have been automatically generated by our software **ZPB** which is described above.

4 Statistics of partially observed diffusions

We have shown in [1] that the Zakai equation provides also a way to compute the likelihood function/ratio in a large variety of statistical problems for partially observed diffusion processes, including : parameter estimation, binary detection, change detection, etc.

An important issue is to prove that these statistical procedures based on the likelihood approach, can provide good estimates or decisions in some asymptotic sense. Consider for example the statistical model

$$dX_t = b_\theta(X_t)\,dt + \varepsilon\,dW_t\ ,$$

$$dY_t = h_\theta(X_t)\,dt + \varepsilon\,dV_t\ ,$$

where $\theta \in \Theta$ is an unknown parameter, which appears in the coefficients $b_\theta(\cdot)$, $h_\theta(\cdot)$ and also in the density $p_0^\theta(\cdot)$ of the initial condition X_0.

Computing the likelihood function for the estimation of the unknown parameter θ on the basis of observations $\{Y_t\,,\,0 \le t \le T\}$, involves the solution of the Zakai equation corresponding to the associated filtering problem. We have proved in [6] the consistency of the MLE under the small noise asymptotics $\varepsilon \downarrow 0$, in the following way

- ◇ using *large deviations* theory, it is proved that the limiting points of the MLE sequence belong to the set of minimizing points of a least-squares type functional for the estimation of θ in the limiting deterministic system $\varepsilon = 0$,
- ◇ under an *identifiability property* of this limiting deterministic system, this set reduces to the "true" value of the parameter.

5 Transfer to the US

F. LeGland has presented some results on time discretization of the Zakai equation [10], and filtering of piecewise linear systems [17], at the IEEE CDC in Tampa (December 1989).

P. Milheiro de Oliveira has presented the results on approximate filters for discrete time systems [13], at the IEEE CDC in Honolulu (December 1990).

E. Pardoux, F. Campillo and F. LeGland have participated to the NSF-INRIA Workshop on Stochastic Analysis, organized at Rutgers University, where the results on particle approximation for first order stochastic PDE [4], and numerical approximation of nonlinear filters and finite time observers [7] have been presented (May 1991).

E. Pardoux and F. LeGland have participated to the International Conference on Stochastic Partial Differential Equations, and have given tutorial lectures at the School-Seminar on Stochastic Partial Differential Equations, organized at the University of Northern Carolina in Charlotte, with partial support of the Army Research Office (May 1991).

E. Pardoux has given a series of lectures on Nonlinear Filtering and Associated Partial Differential Equations, in *Ecole d'Ete de Probabilités XIX* in Saint-Flour (August 1989). The lecture notes [16] present the most recent developpments in the theory of nonlinear filtering, including results obtained by the contractors, for the first time in book form.

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The references [1], [3], [4], [6], [7], [11], [14], [18] and [19] are joined to the report. The reference [12] is included as a part of the Thesis [14], and the reference [15] is included as a part of both Theses [14] and [19].

ZPB

Numerical Experimention in Partially Observed Systems

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Introduction

The purpose of ZPB is to produce Fortran programs for the numerical solution of the Zakai equation. This equation allows to solve various estimation problems in partially observed systems, such as

- state estimation : filtering, fixed–interval smoothing, fixed–lag smoothing, etc.
- detection, either off–line or sequential,
- parameter estimation,
- change detection (disorder), either off-line or sequential.

Given a description of both the model and the problem to be solved, the Fortran programs are generated with the help of the computer algebra system Maple. This description is contained in a file `zpb.data`, in the form of Maple instructions `keyword:=value;`.

The generated Fortran programs rely on routines from the scientific library NAG, which is assumed to be available.

In addition to the Maple program, an interface is provided to help the user defining the model and the problem to be solved, and graphical tools are available to visualize the results. Both the interface and the graphical tools are based on the X Window system.

The possible models and problems to be solved, and some of the algorithms actually implemented in the generated Fortran programs, are presented in these notes.

Maple is a trademark of Waterloo Maple Software. NAG is a trademark of the Numerical Analysis Group Ltd. PostScript is a trademark of Adobe Systems Inc. The X Window system is a trademark of the MIT.

A Models of Partially Observed Systems

STATE

The class of systems to be considered is modeled as a m–dimensional diffusion process

$$dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t \ ,$$

where $\{W_t\,,\,t \geq 0\}$ is a r–dimensional Wiener process with covariance matrix Q. This includes the particular case of systems modeled as the solution of an ordinary differential equation

$$\dot{X}_t = b_t(X_t) \ .$$

> *In the case of a noise driven state equation, it is assumed that the state is one-dimensional. Extension to two-dimensional state is planned.*

OBSERVATIONS

The state of the system is not directly observed. However, d-dimensional noisy nonlinear observations of the state are available, either at discrete times $\{t_1, t_2, \cdots\}$

$$z_k = h_k(X_{t_k}) + v_k \ ,$$

where $\{v_1, v_2, \cdots\}$ is a d-dimensional Gaussian white-noise sequence, with non singular covariance matrix R, or in continuous time

$$z_t = h_t(X_t) + v_t \ ,$$

where $\{v_t\,,\,t \geq 0\}$ is a d-dimensional Gaussian white-noise process, with non singular covariance matrix R. Introducing the integrated observation process

$$Y_t = \int_0^t z_s\,ds \ ,$$

the observation equation becomes

$$dY_t = h_t(X_t)\,dt + dV_t \ ,$$

where $\{V_t\,,\,t \geq 0\}$ is a d-dimensional Wiener process, with non singular covariance matrix R.

Another information about the model is the correlation structure between the state noise and the observation noise.

> *It is assumed that the state noise and the observation noise are independent. Extension to allow noise correlation is planned, see Florchinger–LeGland [4] and [5].*

A first description of the model is provided by the following keywords to be defined in the file **zpb.data**

`dim_state`	dimension m of the state	(integer)
`noise_driven_state`	`true|false`	
	if `noise_driven_state=true`	
`dim_noise`	dimension r of the driving noise	(integer)
`observation_mode`	`discrete|continuous`	
`dim_obs`	dimension d of the observation	(integer)

Coefficients

The next step is to provide the algebraic expression of the drift vector $b_t(\cdot)$, the diffusion matrix $\sigma_t(\cdot)$ (in the case of a noise driven state), and the observation function $h_k(\cdot)$ or $h_t(\cdot)$ depending on whether the observations are available at discrete times or in continuous time. In addition, the probability distribution of the m-dimensional initial state X_0 has to be selected among the following elementary probability distributions

(i) Dirac mass at point x_0,

(ii) Gaussian distribution, with mean μ and covariance matrix Σ,

(iii) uniform distribution on an coordinate cube $[x_1, x_2]$.

Extension to allow mixture of elementary probability distributions is planned.

A further description of the model is provided by the following additional keywords to be defined in the file **zpb.data**

`drift`	drift m-vector	(algebraic Maple expression)
	if `noise_driven_state=true`	
`diffusion`	diffusion (m,r)-matrix	(algebraic Maple expression)
`observation`	observation d-vector	(algebraic Maple expression)
`initial`	`dirac|gaussian|uniform`	

Parameters

The algebraic expression of the coefficients can contain, in addition to the state variable **x**, or **x1,...,xm** if the state is m-dimensional, and the time variable **t**, some other parameters. A list of these parameters is build by the **Maple** program, from the description of the model given in the file **zpb.data**, and stored in the file **.model**. Additional parameters include

(i) parameters of the initial probability distribution,

(ii) covariance matrices Q and R (diagonal) of the noise processes.

The numerical values of all these parameters are read at run time by the generated Fortran programs in two different files **`simul.data`** and **`filt.data`**, This allows to consider mis-specified estimation problems, i.e. to address robustness issues.

> *Extensions to allow the covariances matrices to depend on parameters, and to treat more general robustness issues (different algebraic expression of the coefficients for simulation and filtering), are planned.*

B Estimation Problems and their Solutions

With the diffusion equation

$$dX_t = b_t(X_t)\,dt + \sigma_t(X_t)\,dW_t \ ,$$

is associated the following time dependent second order partial differential operator

$$L_t = \tfrac{1}{2} \sum_{i,j=1}^{m} a_t^{i,j} \, \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_t^i \, \frac{\partial}{\partial x_i} \ ,$$

with the covariance matrix $a_t = \sigma_t \, Q \, \sigma_t^*$.

The state estimation problem consists in estimating the state X_t given only the observations. In the case of discrete time observations, the observation σ–algebra is defined by

$$\mathcal{Z}_k = \sigma(z_1, \cdots, z_k) \ ,$$

whereas, in the case of continuous time observations, it is defined by

$$\mathcal{Y}_t = \sigma(Y_s \, , \, 0 \leq s \leq t) \ .$$

Notation Throughout the paper, the scalar product and the corresponding norm in $\mathbf{R}^d$, associated with the symmetric positive definite matrix R^{-1}, are denoted by $(\,\cdot\,,\,\cdot\,)_{R^{-1}}$ and $|\cdot|_{R^{-1}}$ respectively.

FILTERING

The goal here is to estimate recursively the current state X_t at time t , given only the past observations up to time t.

DISCRETE TIME OBSERVATIONS

Introduce the conditional *a priori* and *a posteriori* probability densities

$$p_k^-(x)\,dx = P(X_{t_k} \in dx \mid \mathcal{Z}_{k-1}) \qquad \text{and} \qquad p_k(x)\,dx = P(X_{t_k} \in dx \mid \mathcal{Z}_k) \ ,$$

respectively. For any $t_{k-1} \leq t \leq t_k$, introduce also

$$p_t^k(x)\,dx = P(X_t \in dx \mid \mathcal{Z}_{k-1}) \ .$$

The transition from $p_{k-1}(x)$ to $p_k(x)$ is divided into two steps

- *prediction step* : between t_{k-1} and t_k , the density $p_t^k(x)$ solves the Fokker–Planck equation

$$\frac{\partial p_t^k}{\partial t} = L_t^* \, p_t^k \ ,$$

with initial condition $p_{t_{k-1}}^k(x) = p_{k-1}(x)$, which gives in particular $p_k^-(x) = p_{t_k}^k(x)$.

- *correction step* : the Bayes formula gives

$$p_k(x) = c_k \cdot \Psi_k(x) \cdot p_k^-(x) \ ,$$

where

$$\Psi_k(x) = \exp\left\{-\tfrac{1}{2}\, |z_k - h_k(x)|^2_{R^{-1}}\right\} \ ,$$

is the *likelihood function* for the estimation of X_{t_k}, based on the observation z_k alone, and c_k is a normalization constant.

Continuous Time Observations

The unnormalized conditional probability density solves the Zakai equation

$$dp_t = L_t^* \, p_t \, dt + p_t \, (h_t, dY_t)_{R^{-1}} \ ,$$

i.e.

$$P(X_t \in dx \mid \mathcal{Y}_t) = c_t \cdot p_t(x) \, dx \ ,$$

where c_t is a normalization constant.

Sampled Observations

Introduce a uniform partition $0 = t_0 < \cdots < t_k < \cdots$ of the time interval $[0, \infty)$, with time step $\Delta = t_k - t_{k-1}$. The first step is to sample the available observation trajectory, i.e. to build the following sequence of compressed observations

$$y_k^\Delta = \frac{1}{\Delta}[Y_{t_k} - Y_{t_{k-1}}] = \frac{1}{\Delta}\int_{t_{k-1}}^{t_k} h_s(X_s)\, ds + \frac{1}{\Delta}[V_{t_k} - V_{t_{k-1}}] \ , \qquad (*)$$

and to use the approximate observation model

$$y_k^\Delta = h_{t_k}(X_{t_k}) + v_k^\Delta \qquad (**)$$

instead, where $\{v_1^\Delta, v_2^\Delta, \cdots\}$ is a Gaussian white noise sequence with covariance matrix R/Δ.

Defining the sampled observation σ–algebra

$$\mathcal{Y}_k^\Delta = \sigma(y_1^\Delta, \cdots, y_k^\Delta) \ ,$$

it is possible to use the results available for the case of discrete time observations. Introduce the conditional *priori* and *a posteriori* probability densities

$$p_k^-(x)\,dx = P(X_{t_k} \in dx \mid \mathcal{Y}_{k-1}^\Delta) \qquad \text{and} \qquad p_k(x)\,dx = P(X_{t_k} \in dx \mid \mathcal{Y}_k^\Delta)\ ,$$

respectively. For any $t_{k-1} \leq t \leq t_k$, introduce also

$$p_t^k(x)\,dx = P(X_t \in dx \mid \mathcal{Y}_{k-1}^\Delta)\ .$$

The transition from $p_{k-1}(x)$ to $p_k(x)$ is divided into two steps just as in the case of discrete time observations, except that the correction step involves

$$\Psi_k^\Delta(x) = \exp\left\{-\tfrac{1}{2}\Delta\ |y_k^\Delta - h_{t_k}(x)|_{R^{-1}}^2\right\}\ ,$$

which is the *likelihood function* for the estimation of X_{t_k} in the approximate observation model (**), based on the sampled observation y_k^Δ alone as defined in (*), and c_k is a normalization constant.

FIXED INTERVAL SMOOTHING

The goal here is to estimate the state X_t at any time $0 \leq t \leq T$, given all the observations in the time interval $[0, T]$.

DISCRETE TIME OBSERVATIONS

Assume that the final time T satisfies $t_N \leq T < t_{N+1}$ for some N. Introduce the conditional smoothing probability density

$$q_k(x)\,dx = P(X_{t_k} \in dx \mid \mathcal{Z}_N)\ .$$

For any $t_{k-1} \leq t \leq t_k$, introduce also

$$q_t^k(x)\,dx = P(X_t \in dx \mid \mathcal{Z}_N)\ .$$

These probability densities are absolutely continuous with respect to the corresponding filtering densities, i.e. $q_k(x) = p_k(x) \cdot v_k(x)$ and $q_t^k(x) = p_t^k(x) \cdot v_t^k(x)$.

The backward transition from $v_k(x)$ to $v_{k-1}(x)$ is divided into two steps

- at time t_k

  $$v_k^-(x) = c_k \cdot \Psi_k(x) \cdot v_k(x)\ ,$$

  where

  $$\Psi_k(x) = \exp\left\{-\tfrac{1}{2}\ |z_k - h_k(x)|_{R^{-1}}^2\right\}\ ,$$

  is again the *likelihood function* for the estimation of X_{t_k}, based on the observation z_k alone, and c_k is a normalization constant.

- between t_k and t_{k-1} , the derivative $v_t^k(x)$ solves the backward Fokker–Planck equation

$$\frac{\partial v_t^k}{\partial t} + L_t\, v_t^k = 0 \ ,$$

with initial condition $v_{t_k}^k(x) = v_k^-(x)$, which gives in particular $v_{k-1}(x) = v_{t_{k-1}}^k(x)$.

It is immediate by duality, that

$$(p_k, v_k) = (p_k^-, v_k^-) = (p_{k-1}, v_{k-1}) \ ,$$

which implies that the conditional densities $q_k(x)$ are properly normalized.

Continuous Time Observations

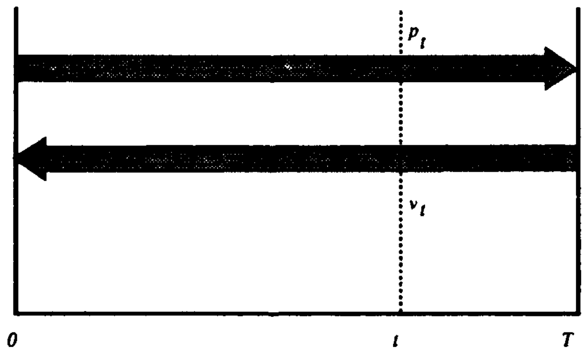
Here also, the unnormalized smoothing conditional probability density is absolutely continuous with respect to the corresponding unnormalized filtering conditional probability density, i.e. $q_t(x) = p_t(x) \cdot v_t(x)$ where the derivative solves the backward Zakai equation

$$dv_t + L_t\, v_t\, dt + v_t\, (h_t, dY_t)_{R^{-1}} = 0 \ ,$$

i.e.

$$P(X_t \in dx \mid \mathcal{Y}_T) = c_T \cdot q_t(x)\, dx \ ,$$

where c_T is a normalization constant.



Sampled Observations

This case is very similar to the case of discrete time observations. Introduce the conditional smoothing probability density

$$q_k(x)\, dx = P(X_{t_k} \in dx \mid \mathcal{Y}_N^\Delta) \ .$$

For any $t_{k-1} \leq t \leq t_k$, introduce also

$$q_t^k(x)\,dx = P(X_t \in dx \mid \mathcal{Y}_N^\Delta)\ .$$

These probability densities are absolutely continuous with respect to the corresponding filtering densities, i.e. $q_k(x) = p_k(x) \cdot v_k(x)$ and $q_t^k(x) = p_t^k(x) \cdot v_t^k(x)$.

The backward transition from $v_k(x)$ to $v_{k-1}(x)$ is divided into two steps, just as in the case of discrete time observations, except that the correction step involves

$$\Psi_k^\Delta(x) = \exp\left\{-\tfrac{1}{2}\Delta\ |y_k^\Delta - h_{t_k}(x)|_{R^{-1}}^2\right\}\ ,$$

which is the *likelihood function* for the estimation of X_{t_k} in the approximate observation model (**), based on the sampled observation y_k^Δ alone as defined in (*), and c_k is a normalization constant.

The description of the estimation problem to be solved is provided by the following keyword to be defined in the file `zpb.data`

```
problem     filtering | smoothing
```

Extensions to other state estimation problems, such as fixed-lag smoothing, or to statistical problems, including parameter estimation, detection, change detection, etc., either off-line or sequential, are planned, see Campillo–LeGland [1].

C Numerical Algorithms

Given a model and an estimation problem to be solved about this model, both described in the **Maple** file **zpb.data**, the purpose of **ZPB** is to provide **Fortran** programs and visualization tools, for the numerical experimentation and evaluation of the estimation algorithm. This involves two different tasks

- *simulation* : the goal here is to generate a trajectory of the state process, and a sequence of either discrete time or sampled observations, to be stored into the files **zpb.state** and **zpb.obs** respectively.
- *estimation* : the goal here is to combine *a priori* information about the model and the observations to be read from the file **zpb.obs**, in order to solve the selected estimation problem. Results are stored into the files **zpb.estim** and **zpb.density**.

From now on, the state process is assumed to be one-dimensional.

Extensions to allow two-dimensional state process, are planned.

SIMULATION

The time horizon T and the time step Δ between successive observations. are read from the file **algo.data**, under the name **tmax** and **dt** respectively. A refined time step $\Delta_M = \Delta/M$ is introduced, where the number M of local iterations for the simulation is also read from the file **algo.data**, under the name **locsimul**.

The state X_{t_k} is approximated by $\bar{x}_k$ using the Milshtein discretization scheme [10]

$$\left\{\begin{aligned} t_k^0 &= t_{k-1} \;, \\ t_k^i &= t_k^{i-1} + \Delta_M \;, \qquad 1 \leq i \leq M \\ t_k &= t_k^M \;, \end{aligned}\right.$$

$$\left\{\begin{aligned} x_k^0 &= \bar{x}_{k-1} \;, \\ x_k^i &= x_k^{i-1} + b_{t_k^{i-1}}(x_k^{i-1})\, \Delta_M + \sigma_{t_k^{i-1}}(x_k^{i-1})\, w_k^i \\ &\qquad + \tfrac{1}{2}\, \sigma_{t_k^{i-1}}(x_k^{i-1})\, \sigma'_{t_k^{i-1}}(x_k^{i-1})\, [\, |w_k^i|^2 - \Delta_M] \;, \qquad 1 \leq i \leq M \\ \bar{x}_k &= x_k^M \;, \end{aligned}\right.$$

where $\{w_k^1, \cdots, w_k^M\}$ is a Gaussian white-noise sequence with covariance matrix $Q \cdot \Delta_M$.

On the other hand, the generation of the sequence of observations is different whether they are discrete time or sampled observations.

DISCRETE TIME OBSERVATIONS

The observation z_k is simulated using the approximation $\bar{x}_k$ of the state X_{t_k}

$$z_k = h_k(\bar{x}_k) + v_k \ ,$$

where v_k is a Gaussian random vector with non singular covariance matrix R.

SAMPLED OBSERVATIONS

The sampled observation y_k^Δ as defined in $(*)$ are simulated using the approximation $\{x_k^1, \cdots, x_k^M\}$ of the state process between t_{k-1} and t_k

$$y_k^\Delta = \frac{1}{M} \sum_{i=1}^{M} h_{t_k^i}(x_k^i) + v_k^\Delta \ ,$$

where v_k^Δ is a Gaussian random vector with non singular covariance matrix R/Δ.

Gaussian random variables are generated according to the Box-Muller algorithm, see Rubinstein [12]. A routine `boxmuller` is provided in the library `libzpb.a`. Uniformly distributed random variables are generated by the NAG library routine `g05caf`, in conjunction with `g05cbf`. The seed of the random generator is read from the file `algo.data` under the name `seed`.

Extensions to include more accurate discretization of the deterministic part of the state equation, are planned.

ESTIMATION

It follows from the discussion above, that either for the filtering or the smoothing problem, and whether the observations are in discrete time or sampled, it is needed to discretize the Fokker-Planck equation between t_{k-1} and t_k

$$\frac{\partial p_t^k}{\partial t} = L_t^* \, p_t^k \ .$$

Time Discretization

The time horizon T and the time step Δ between successive observations, are read from the file **algo.data**, under the name **tmax** and **dt** respectively. A refined time step $\Delta_P = \Delta/P$ is introduced, where the number P of local iterations for the prediction step is also read from the file **algo.data**, under the name **locpred**.

The filtering density p_k is approximated by $\bar{p}_k$ using the implicit Euler scheme

$$\left\{\begin{array}{ll} t_k^0 = t_{k-1} \ , & \\ t_k^i = t_k^{i-1} + \Delta_P \ , & 1 \leq i \leq P \\ t_k = t_k^P \ , & \end{array}\right.$$

$$\left\{\begin{array}{rll} p_k^0 &= \bar{p}_{k-1} \ , & \\ [I - \Delta_P \, L_{t_k^{i-1}}]^* \, p_k^i &= p_k^{i-1} \ , & 1 \leq i \leq P \\ \bar{p}_k &= c_k \cdot \Psi_k \cdot p_k^P \ . & \end{array}\right.$$

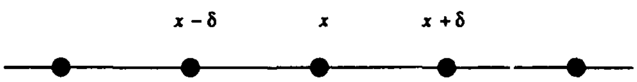
This approximation is based on

$$p_{t_k^i} - p_{t_k^{i-1}} = \int_{t_k^{i-1}}^{t_k^i} L_s^* \, p_s \, ds \simeq \int_{t_k^{i-1}}^{t_k^i} L_s^* \, ds \, p_{t_k^i} \simeq \Delta_P \, L_{t_k^{i-1}}^* \, p_{t_k^i} \ .$$

Space Discretization

Following Kushner [8], the second order partial differential operator L_t is approximated by a finite difference matrix, on a regular bounded coordinate grid. Only the one–dimensional case is considered here. The bounded domain is an interval $\bar{D} = [\underline{x}, \overline{x}]$. The end points $\underline{x}$, $\overline{x}$ and the mesh size δ are read from the file **algo.data** under the name **xmin**, **xmax** and **dx** respectively.

Let $\mathbf{R}_\delta$ denote a one–dimensional coordinate grid with mesh size δ. For any $x \in \mathbf{R}_\delta$, let $N_\delta(x) = \{x, x \pm \delta\}$ denote the set of neighbours. The restricted grid is $\bar{D}_\delta = \bar{D} \cap \mathbf{R}_\delta$, the set of interior grid points is $D_\delta = \{x \in \bar{D}_\delta \ : \ N_\delta(x) \subset \bar{D}_\delta\}$, and the set of boundary points is $\Gamma_\delta = \bar{D}_\delta \setminus D_\delta$.



The first order partial derivative is approximated by non centered *upwind* finite difference schemes, in a way depending on the drift coefficient

$$\frac{\partial \phi}{\partial x}(x) \simeq \begin{cases} \dfrac{\phi(x+\delta)-\phi(x)}{\delta} \;, & \text{if } b_t(x) \geq 0 \\[2ex] \dfrac{\phi(x)-\phi(x-\delta)}{\delta} \;, & \text{if } b_t(x) < 0 \end{cases}$$

The second order partial derivative is approximated by the usual centered finite difference scheme

$$\frac{\partial^2 \phi}{\partial x^2}(x) \simeq \frac{\phi(x+\delta) - 2\phi(x) + \phi(x-\delta)}{\delta^2} \;.$$

It follows that for any interior point $x \in D_\delta$

$$\begin{aligned} L_t\, \phi(x) &\simeq b_t^+(x)\, \frac{\phi(x+\delta)-\phi(x)}{\delta} + b_t^-(x)\, \frac{\phi(x)-\phi(x-\delta)}{\delta} \\ &\quad + \tfrac{1}{2} a_t(x)\, \frac{\phi(x_\delta) - 2\phi(x) + \phi(x-\delta)}{\delta^2} \\ &\simeq L_t^\delta\, \phi(x) = \sum_{y \in \bar{D}_\delta} L_t^\delta(x,y)\, \phi(y) \;. \end{aligned}$$

The only non zero terms in the time dependent matrix L_t^δ are the terms involving neighbours

$$\begin{cases} L_t^\delta(x, x \pm \delta) = \dfrac{1}{\delta^2}\, \left[\tfrac{1}{2}\, a_t(x) + \delta\, b_t^\pm(x) \right] \\[2ex] L_t^\delta(x,x) = -\dfrac{1}{\delta^2}\, \left[a_t(x) + \delta\, |b_t(x)| \right] \end{cases}$$

and in addition the following properties are satisfied

$$L_t^\delta(x,y) \geq 0 \qquad \text{for } y \neq x$$

$$\sum_{y \in \bar{D}_\delta} L_t^\delta(x,y) = 0$$

$$\lambda_t^\delta(x) = -L_t^\delta(x,x) = \sum_{y \neq x} L_t^\delta(x,y) \geq 0$$

where the latter is a consequence of the two others.

For any boundary point $x \in \Gamma_\delta$, the definition of the finite difference matrix depends on the boundary condition to be satisfied.

- ***absorbing (stopping)*** boundary :

$$L_t^\delta(x,y) = 0 \qquad \text{for all } y \in N_\delta(x) \;,$$

- *reflecting* boundary :

  at leftmost grid point

$$\begin{cases} L_t^\delta(x,x) = -\lambda_t^\delta(x) \\ L_t^\delta(x,x+\delta) = \lambda_t^\delta(x) \\ L_t^\delta(x,x-\delta) = 0\ , \end{cases}$$

  at rightmost grid point

$$\begin{cases} L_t^\delta(x,x) = -\lambda_t^\delta(x) \\ L_t^\delta(x,x+\delta) = 0 \\ L_t^\delta(x,x-\delta) = \lambda_t^\delta(x)\ . \end{cases}$$

The nature of the boundary condition is read from the file `algo.data` under the name `boundary` which can take the two possible values `stopping|reflection`.

These properties show that the time dependent matrix L_t^δ is the instantaneous jump intensity matrix of a pure jump Markov process $\{X_t^\delta\ , t \geq 0\}$ evolving on the grid $\bar{D}_\delta$. Conditionned on the current position $x \in \bar{D}_\delta$, the inter–jump time and the next position are independent random variables. Moreover, the inter–jump time is an exponential random variable with parameter $\lambda_\delta(x)$, and the probability distribution of the next position is given by $\pi_\delta(x,y) = L_\delta(x,y)/\lambda_\delta(x)$, for all $y \in \bar{D}_\delta$.

FULL DISCRETIZATION

Combining finite difference approximation of the second order partial differential operator L_t, with implicit Euler time discretization, results in the following sequence of linear systems

$$\begin{cases} p_k^0 = \bar{p}_{k-1}\ , \\ [I - \Delta_P\, L_{t_k^{i-1}}^\delta]^*\, p_k^i = p_k^{i-1}\ , \qquad 1 \leq i \leq P \\ \bar{p}_k = c_k \cdot \Psi_k \cdot p_k^P\ . \end{cases}$$

This is a tridiagonal linear system which can be solved by direct method : first, the matrix is factorized using Gaussian elimination algorithm with partial pivoting, then the resulting upper and lower triangular systems are solved. This is done by the NAG library routines **`f01lef`** and **`f04lef`** respectively.

> ***Extensions to include iterative methods, or multigrid methods for the solution of the linear system are planned, and will have to be used in the case of a multi-dimensional state equation.***

PARAMETERS

A list of the algorithm parameters is build by the **Maple** program, and stored in the file `.algo`. These parameters include

dt — time step,

tmin, **tmax** — ends of the time interval,

locsimul — number of local iterations for the simulation,

seed — seed of the random number generator,

xmin, **xmax** — bounds of the bounded space discretization grid,

dx — mesh of the space discretization grid,

locpred — number of local iterations for the prediction,

boundary — nature of the boundary condition, etc.

D Organization of a Session

The user has first to provide a description of both the model and the problem to be solved. This description is contained in a file `zpb.data`, in the form of Maple instructions **`keyword:=value;`**. The list of admissible keywords is summarized below

`dim_state`	dimension m of the state	(integer)
`noise_driven_state`	`true|false`	
	if `noise_driven_state=true`	
`dim_noise`	dimension r of the driving noise	(integer)
`observation_mode`	`discrete|continuous`	
`dim_obs`	dimension d of the observation	(integer)
`drift`	drift m–vector	(Maple expression)
	if `noise_driven_state=true`	
`diffusion`	diffusion (m,r)–matrix	(Maple expression)
`observation`	observation d–vector	(Maple expression)
`initial`	`dirac|gaussian|uniform`	
`problem`	`filtering	smoothing`	

The file `zpb.data` can be created with any text editor. Under X Window, a user interface is provided to create and modify the file `zpb.data` automatically.

When this first stage is completed, the Maple program is called, and Fortran programs are generated and compiled. Two data files `.model` and `.algo` are also generated, which contain respectively the list of parameters relevant to the model and the problem to be solved. The user has to provide numerical values for these parameters, which are contained in three different input files `simul.data`, `estim.data` and `algo.data`. Here again, these files can be created and modified with any text editor. Under X Window, the user interface allows to create and modify the files `simul.data`, `estim.data` and `algo.data` automatically.

When this second stage is completed, the Fortran files are executed and the results are stored in the output files `zpb.state`, `zpb.estim` and `zpb.density`, which contain the simulated state sequence, the estimated state sequence (usually the conditional mean), and the conditional density sequence.

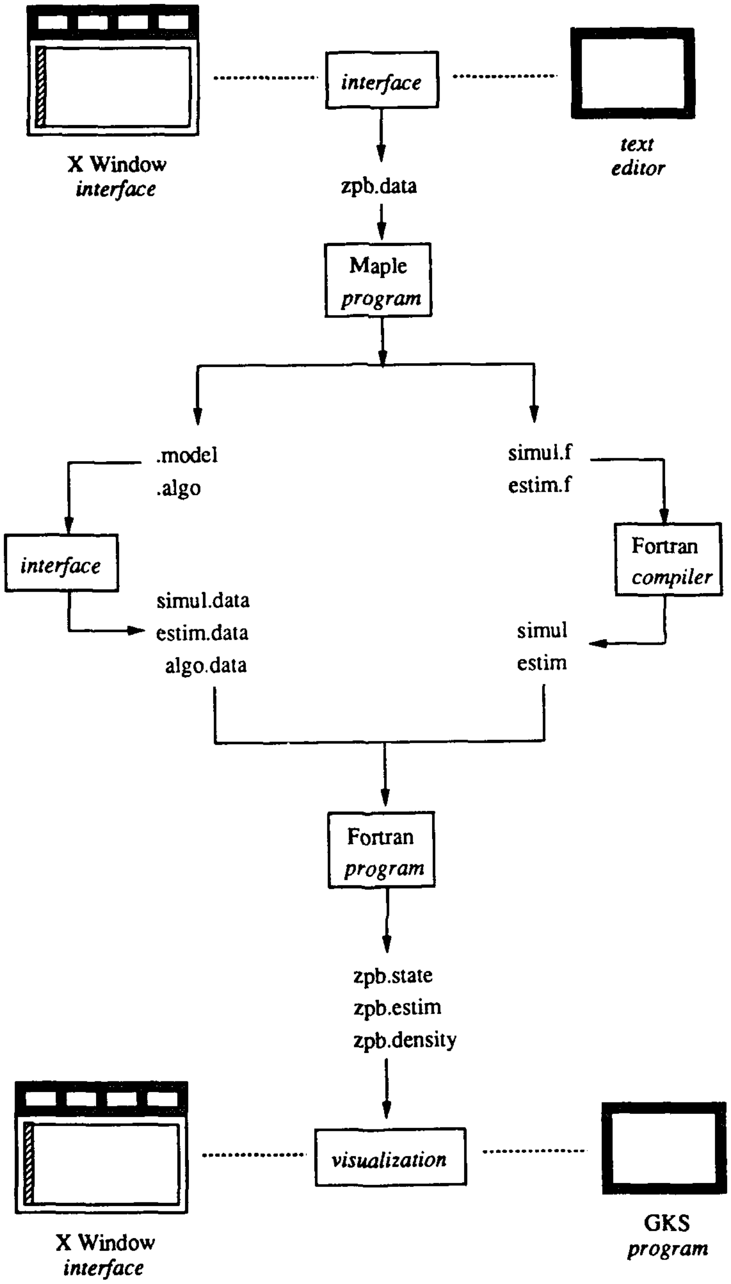
Under X Window, graphical tools are provided, which allow to visualize the data in the output files, in the following way

- a representation of both the simulated state sequence and the estimated state sequence vs. time, is provided in a first window, as well as a shaded area representing some confidence region for the conditional distribution at a given level,
- a time can be selected in the first window, and a representation of the conditional density at the selected time is then provided in a second window,

- it is also possible to visualize in the second window, the continuous time evolution of the conditional density.

In earlier versions of the software, Fortran programs were provided for the graphic visualisation of the results, based on the Graphical Kernel System GKS [3,7] library.

The organization of a ZPB session is summarized by the following flowchart.


interface
X Window
interface
text
editor
zpb.data
Maple
program
.model
.algo
simul.f
estim.f
interface
Fortran
compiler
simul.data
estim.data
algo.data
simul
estim
Fortran
program
zpb.state
zpb.estim
zpb.density
visualization
X Window
interface
GKS
program

E Example

Consider the nonlinear filtering problem for the following quadratic sensor system

$$\begin{cases} dX_t = -\beta X_t\, dt + dW_t\ , \\ \\ dY_t = (X_t + \varepsilon X_t^2)\, dt + dV_t\ , \end{cases}$$

where $\{W_t\, ,\, t \geq 0\}$ and $\{V_t\, ,\, t \geq 0\}$ are independent Wiener processes with variance Q and R respectively. The initial condition X_0 is a Gaussian random variable with mean μ and variance Σ. Note that when $\varepsilon = 0$, the observation function is linear and the conditional law is Gaussian, whereas when $\varepsilon \neq 0$, the observation function is symmetric around $x_0 = -1/2\varepsilon$, and is not injective, which can result in a multi-modal conditional density when the signal is in the neighbourhood of x_0.

This model is described by a sequence of Maple instructions in the file **zpb.data**. The corresponding Fortran files are automaticaly generated and compiled, as well as the file **.model** containing the list of model parameters, see Figure 1.

Two numerical examples are considered below, for different values of the parameter ε. Numerical values of the model parameters are given in the following table

β	0.2
ε	*0.0 or* 0.25
μ	0.0
Σ	1.0
Q	1.0
R	0.001

In both Figure 2 and 3, the level of the shaded confidence region is 0.95

```
observation_mode:='continuous';
noise_driven_state:=true;
dim_state:=1;
initial:='gaussian';
drift:=-beta*x;
dim_noise:=1;
diffusion:=1;
dim_obs:=1;
observation:=x+eps*x^2;
problem:='filtering';
```

```
beta
eps
x0
q0
qq
rr
```

Figure 1: Description file `zpb.data` and corresponding parameter file `.model`

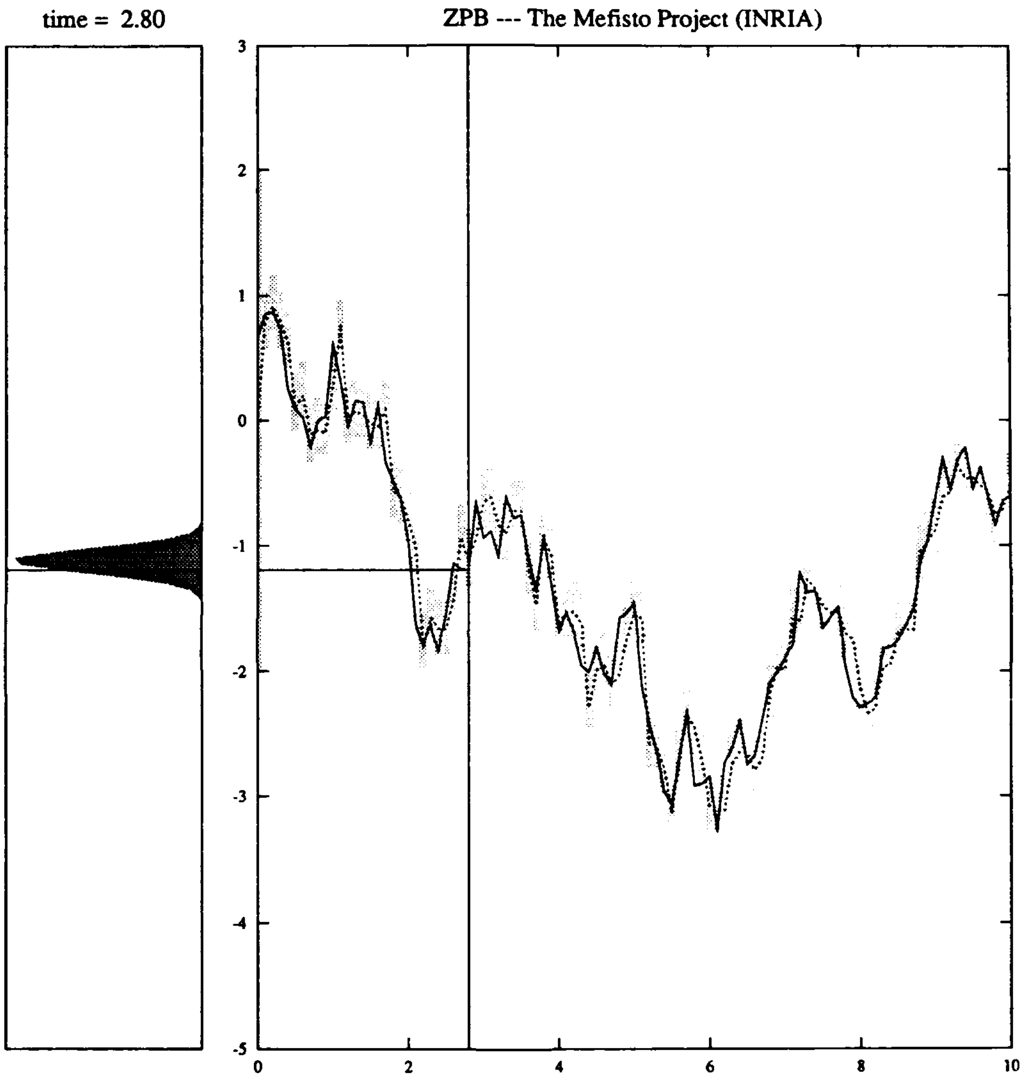


Figure 2: Linear observation function ($\epsilon = 0.00$)

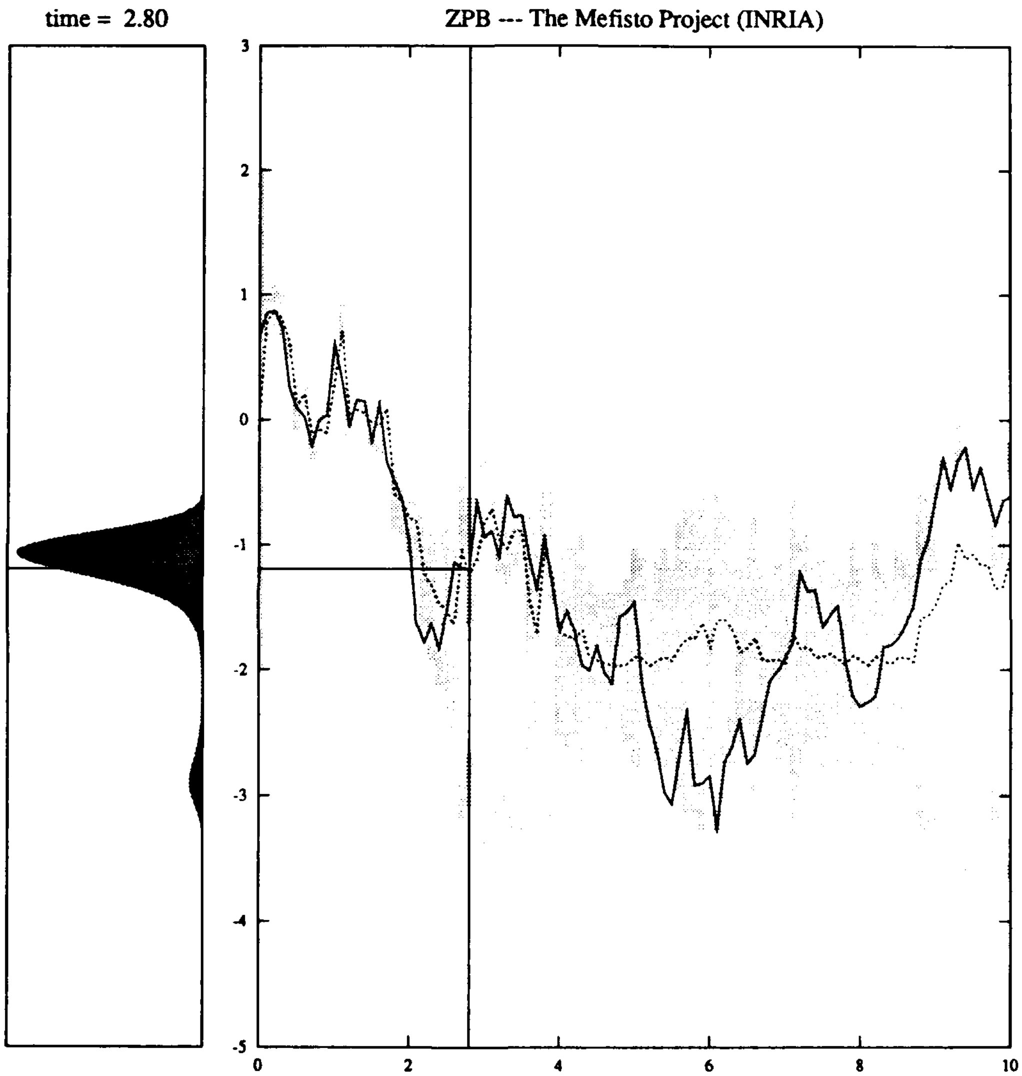


Figure 3: Non-injective observation function ($\varepsilon = 0.25$)

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Antoine Gondel (ENST–Paris) has contributed to the first Macsyma version of ZPB, see [6]. Toufic Abboud (Ecole Polytechnique–Palaiseau) has written the following Reduce/Le_Lisp version of ZPB. These two versions are no longer in use. Rivo Rakotozafy (ESSI–Sophia Antipolis) has written the extended Kalman filter module, Marc Joannidès (Université de Provence–Marseille) has written the particle method module for noise free state systems, in the current Maple version. Robert Fournier (INRIA–Sophia Antipolis) has provided the GKS driver for X Window workstation, and additional visualization tools. Fabrice Rouillier (Université de Rennes) has written the user interface and the visualization tools under X Window.

Michiel Hazewinkel (CWI–Amsterdam), Ben–Zion Bobrovsky (Tel–Aviv University and ETH–Zürich), Ben Hanzon (Vrije Universiteit–Amsterdam), Boris Rozovskii (University of Southern California–Los Angeles) have shown interest in earlier versions of ZPB.

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TIME DISCRETIZATION OF NONLINEAR FILTERING EQUATIONS*

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Abstract *Some computable approximate expressions are provided for the conditional law of diffusion processes observed in continuous time. The numerical schemes are derived through an approximation of the original filtering problem. Given a partition of the time interval, this procedure consists in sampling the available observation sample path and approximating the a priori law of the diffusion process. This results in approximation schemes for the Zakai equation, for which rate of convergence are provided.*

1 Introduction

The purpose of this paper is to give computable and accurate approximate expressions for the conditional law of a diffusion process observed in continuous time. Since this conditional law depends on both

- the a priori information, provided by the semi-group $\{P_t\,,\,t \geq 0\}$ or equivalently the infinitesimal generator L,
- the available observation sample-path $\{Y_t\,,\,t \geq 0\}$.

the approximation problem under consideration should reduce in some sense to

- approximate the a priori law of the original diffusion process, e.g. by the more simple a priori law of some other process,
- extract the most useful information from the available continuous time measurements $\{Y_t\,,\,t \geq 0\}$.

The general situation of filtering a signal process from noisy continuous measurements will be considered. At each step of the approximation procedure, the general formulas will be applied to the particular case of diffusion processes, in order to check whether or not some computable expression has been obtained. Note that only time discretization is considered here: the discretization with respect to the space variable, e.g. the approximation of the partial differential operator L by finite differences is not taken into consideration.

*Research partially supported by Systems Research Center, University of Maryland, and by USACCE under Contract DAJA45-87-M-0296

2 The filtering problem

On a measurable space $(\Omega, \mathcal{F})$ are given a probability measure P, and a pair of stochastic processes $\{X_t\,,\,t \geq 0\}$ and $\{Y_t\,,\,t \geq 0\}$ taking values in $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively, such that under P

$$dY_t = h(X_t)\,dt + dV_t\,, \tag{1}$$

where $\{V_t\,,\,t \geq 0\}$ is a standard Wiener process, independent of $\{X_t\,,\,t \geq 0\}$.

Note that the a priori law of the signal $\{X_t\,,\,t \geq 0\}$ is not specified at this point. The observation function satisfy the following hypothesis

h is a measurable and bounded function from $\mathbf{R}^m$ to $\mathbf{R}^d$.

Remark 2.1 As usual, (1) is the mathematical way of expressing that some measurement

$$z_t = h(X_t) + \eta_t\,, \tag{2}$$

is available at time t, where $\{\eta_t\,,\,t \geq 0\}$ is a Gaussian white-noise process, independent of $\{X_t\,,\,t \geq 0\}$.

Introduce the σ-algebras

$$\mathcal{F}_t \stackrel{\Delta}{=} \sigma(X_s\,,\,0 \leq s \leq t)\,,$$

$$\mathcal{Y}_t^s \stackrel{\Delta}{=} \sigma(Y_\tau - Y_s\,,\,s \leq \tau \leq t)\,, \quad \mathcal{Y}_t \stackrel{\Delta}{=} \mathcal{Y}_t^0\,.$$

The problem is to estimate X_t from $\mathcal{Y}_t$, i.e. to compute the conditional (a posteriori) law of X_t given $\mathcal{Y}_t$, defined by $\mathbf{E}(\phi(X_t) \mid \mathcal{Y}_t)$. Introducing

$$Z_t^s \stackrel{\Delta}{=} \exp\left\{\int_s^t h^*(X_\tau)\,dY_\tau - \tfrac{1}{2}\int_s^t |h(X_\tau)|^2\,d\tau\right\}\,,$$

and $Z_t \stackrel{\Delta}{=} Z_t^0$, it is standard that, for all $T > 0$ the original probability measure P is equivalent on $[0,T]$ to the *reference probability* measure $P^\dagger$ with Radon-Nikodym derivative Z_T, so that under $P^\dagger$ $\{Y_t\,,\,t \geq 0\}$ is a standard Wiener process, independent of $\{X_t\,,\,t \geq 0\}$.

By the Bayes formula

$$\mathbf{E}(\phi(X_t) \mid \mathcal{Y}_t) = \frac{\mathbf{E}^\dagger(\phi(X_t)Z_t \mid \mathcal{Y}_t)}{\mathbf{E}^\dagger(Z_t \mid \mathcal{Y}_t)}\,,$$

so that it is enough to compute $\{p_t, t \geq 0\}$ defined by

$$(p_t, \phi) \triangleq \mathbf{E}^\dagger(\phi(X_t) Z_t \mid \mathcal{Y}_t) .$$

In the particular case where the signal $\{X_t, t \geq 0\}$ is a diffusion process with infinitesimal generator L, $\{p_t, t \geq 0\}$ is the unique solution of the Zakai equation

$$dp_t = L^* p_t \, dt + h^* p_t \, dY_t . \tag{3}$$

It is readily seen on this equation that p_t depends on the a priori law of $\{X_t, t \geq 0\}$ represented by the partial differential operator L, and on the observation sample-path $\{Y_t, t \geq 0\}$. However, equation (3) is not computable and should be approximated. The approach presented here, is to rather approximate the original filtering problem by a simpler problem, and to consider the resulting equation for the conditional law in this new filtering problem as an approximation to equation (3). In Section 5, the rate of convergence for such approximations will be provided, by direct numerical analysis of equation (3).

The presentation adopted follows Korezlioglu-Mazziotto [2]. There is indeed three successive steps in the global approximation procedure. In the first step, sampling and data compression of the observation sample-path $\{Y_t, t \geq 0\}$ is performed. Then, the signal $\{X_t, t \geq 0\}$ is approximated by some piecewise constant process $\{\overline{X}_t, t \geq 0\}$. In the last step, the a priori law of the process $\{\overline{X}_t, t \geq 0\}$ is approximated. Only the first two steps will be considered here.

3 Sampling of the observation sample–path

Throughout the paper, an infinite partition

$$0 = t_0 < t_1 < \cdots < t_n < \cdots$$

of $[0, +\infty)$ is introduced, to be denoted by π, with time increments $\delta_i \triangleq t_{i+1} - t_i$.

Sampling and data compression is the pre-processing procedure by which the new information contained in the continuous measurements received in the time interval $t_i \leq t \leq t_{i+1}$ and represented by $\mathcal{Y}^{t_i}_{t_{i+1}}$, is summarized into a finite number of random variables. This is formalized in the following

Definition 3.1 *An admissible sampling procedure relative to the partition π is a family $\{\overline{\mathcal{Y}}^{t_i}_{t_{i+1}}, i \geq 0\}$ of σ-algebras which satisfy, for all $i \geq 0$*

(i) $\overline{\mathcal{Y}}^{t_i}_{t_{i+1}}$ is generated by a finite number of random variables,

(ii) $\overline{\mathcal{Y}}^{t_i}_{t_{i+1}} \subset \mathcal{Y}^{t_i}_{t_{i+1}}$.

In addition, the following notations are used

$$\overline{\mathcal{Y}}^{t_l}_{t_m} \triangleq \bigvee_{i=l}^{m-1} \overline{\mathcal{Y}}^{t_i}_{t_{i+1}} , \quad \overline{\mathcal{Y}}_{t_m} \triangleq \overline{\mathcal{Y}}^{0}_{t_m} .$$

Here are two examples of admissible sampling procedures, to be considered throughout the paper.

Example 1. Define

$$\xi_i \triangleq Y_{t_{i+1}} - Y_{t_i} = \int_{t_i}^{t_{i+1}} z_s \, ds , \tag{4}$$

which is the mean value of the actual measurements (2) on the time interval $t_i \leq s \leq t_{i+1}$. In this example, $\overline{\mathcal{Y}}^{t_i}_{t_{i+1}}$ is generated by the random variable ξ_i. Note that, under the reference probability measure $P^\dagger$, $\{\xi_i, i \geq 0\}$ are mutually independent d-dimensional Gaussian random variables with zero mean and covariance matrix $\delta_i I$.

Example 2. Define

$$\xi^\sharp_i \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} (s - t_i) \, dY_s = \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \int_{s}^{t_{i+1}} y_\tau \, d\tau \, ds ,$$

$$\xi^\flat_i \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} (t_{i+1} - s) \, dY_s = \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} y_\tau \, d\tau \, ds ,$$

which are two other different ways of computing some mean value of the actual measurements (2) on the time interval $t_i \leq s \leq t_{i+1}$. In this example, $\overline{\mathcal{Y}}^{t_i}_{t_{i+1}}$ is generated by the random variables $\xi^\sharp_i$ and $\xi^\flat_i$. Note that $\xi^\sharp_i + \xi^\flat_i = \xi_i$ and that, under the reference probability measure $P^\dagger$, $\{(\xi^\sharp_i, \xi^\flat_i), i \geq 0\}$ are mutually independent $2d$-dimensional Gaussian random variables with zero mean and covariance matrix $\delta_i \Sigma$, where

$$\Sigma = \begin{pmatrix} \frac{1}{3} I & \frac{1}{6} I \\ \frac{1}{6} I & \frac{1}{3} I \end{pmatrix} .$$

In particular, the characteristic function of $(\xi^\sharp_i, \xi^\flat_i)$ satisfies

$$\begin{aligned} \chi(a, b) &\triangleq \mathbf{E}^\dagger(\exp\left\{a^* \xi^\sharp_i + b^* \xi^\flat_i\right\}) \\ &= \exp\left\{\tfrac{1}{6}(|a|^2 + a^* b + |b|^2)\delta_i\right\} . \end{aligned} \tag{5}$$

The problem is now to estimate X_{t_i} from $\overline{\mathcal{Y}}_{t_i}$, i.e. to compute the conditional law of X_{t_i} given $\overline{\mathcal{Y}}_{t_i}$. By the Bayes formula

$$\mathbf{E}(\phi(X_{t_i}) \mid \overline{\mathcal{Y}}_{t_i}) = \frac{\mathbf{E}^\dagger(\phi(X_{t_i}) Z_{t_i} \mid \overline{\mathcal{Y}}_{t_i})}{\mathbf{E}^\dagger(Z_{t_i} \mid \overline{\mathcal{Y}}_{t_i})} ,$$

so that it is enough to compute $\{p_i, i \geq 0\}$ defined by

$$(p_i, \phi) \triangleq \mathbf{E}^\dagger(\phi(X_{t_i}) Z_{t_i} \mid \overline{\mathcal{Y}}_{t_i}) .$$

The first step is provided by the following

Proposition 3.2 *Introduce*

$$\Xi^{t_i}_{t_{i+1}} \triangleq \mathbf{E}^\dagger(Z^{t_i}_{t_{i+1}} \mid \mathcal{F}_{t_{i+1}} \vee \overline{\mathcal{Y}}^{t_i}_{t_{i+1}}) ,$$

$$U_{i+1}\phi \triangleq \mathbf{E}^\dagger(\phi(X_{t_{i+1}}) \Xi^{t_i}_{t_{i+1}} \mid \mathcal{F}_{t_i} \vee \overline{\mathcal{Y}}^{t_i}_{t_{i+1}}) .$$

Then

$$(p_{i+1}, \phi) = \mathbf{E}^\dagger([U_{i+1}\phi] Z_{t_i} \mid \overline{\mathcal{Y}}_{t_{i+1}}) . \tag{6}$$

PROOF.

$$
\begin{aligned}
&(p_{i+1}, \phi) \\
&= \mathbf{E}^\dagger(\phi(X_{t_{i+1}}) Z_{t_i} Z^{t_i}_{t_{i+1}} \mid \overline{\mathcal{Y}}_{t_{i+1}}) \\
&= \mathbf{E}^\dagger(\phi(X_{t_{i+1}}) Z_{t_i} \\
&\qquad \cdot \mathbf{E}^\dagger(Z^{t_i}_{t_{i+1}} \mid \mathcal{F}_{t_{i+1}} \vee \mathcal{Y}_{t_i} \vee \overline{\mathcal{Y}}^{t_i}_{t_{i+1}}) \mid \overline{\mathcal{Y}}_{t_{i+1}}) \\
&= \mathbf{E}^\dagger(\phi(X_{t_{i+1}}) Z_{t_i} \Xi^{t_i}_{t_{i+1}} \mid \overline{\mathcal{Y}}_{t_{i+1}}) \\
&= \mathbf{E}^\dagger(Z_{t_i} \mathbf{E}^\dagger(\phi(X_{t_{i+1}}) \Xi^{t_i}_{t_{i+1}} \mid \mathcal{F}_{t_i} \vee \mathcal{Y}_{t_i} \vee \overline{\mathcal{Y}}^{t_i}_{t_{i+1}}) \mid \overline{\mathcal{Y}}_{t_{i+1}}) \\
&= \mathbf{E}^\dagger([U_{i+1}\phi] Z_{t_i} \mid \overline{\mathcal{Y}}_{t_{i+1}}) . \qquad \square
\end{aligned}
$$

Going back to the examples introduced above, the expression for $\Xi^{t_i}_{t_{i+1}}$ will be derived, and it will be checked whether or not the additional hypothesis that the signal $\{X_t, t \geq 0\}$ is a diffusion process can lead to computable expressions.

Example 1 (Continued). For the sampling procedure defined by ξ_i, it is proved in [2] that

$$
\Xi^{t_i}_{t_{i+1}} = \exp\left\{ h_i^* \xi_i - \tfrac{1}{2} |h_i|^2 \delta_i \right\} , \tag{7}
$$

where

$$
h_i \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} h(X_s)\, ds .
$$

However, replacing this expression into (6) does not provide a computable expression, even if the additional hypothesis that the signal $\{X_t, t \geq 0\}$ is a diffusion process is introduced.

Example 2 (Continued). For the sampling procedure defined by $(\xi_i^\sharp, \xi_i^\flat)$, it can be proved that

$$
\begin{aligned}
\Xi^{t_i}_{t_{i+1}} &= \exp\left\{ [h_i^\sharp]^* \xi_i^\sharp + [h_i^\flat]^* \xi_i^\flat \right. \\
&\qquad \left. - \tfrac{1}{6}(|h_i^\sharp|^2 + [h_i^\sharp]^* h_i^\flat + |h_i^\flat|^2)\delta_i \right\} \\
&= \exp\left\{ [h_i^\sharp]^* \xi_i^\sharp - \tfrac{1}{4}|h_i^\sharp|^2 \delta_i \right\} \\
&\quad \cdot \exp\left\{ [h_i^\flat]^* \xi_i^\flat - \tfrac{1}{4}|h_i^\flat|^2 \delta_i \right\} \cdot \exp\left\{ \tfrac{1}{12}|h_i^\sharp - h_i^\flat|^2 \delta_i \right\}
\end{aligned} \tag{8}
$$

where

$$
h_i^\sharp \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \overline{w}\Big(\frac{s - t_i}{t_{i+1} - t_i}\Big) h(X_s)\, ds ,
$$

$$
h_i^\flat \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \overline{w}\Big(\frac{t_{i+1} - s}{t_{i+1} - t_i}\Big) h(X_s)\, ds ,
$$

and the weight function $\overline{w}$ is defined for all $0 \leq \theta \leq 1$ by $\overline{w}(\theta) \triangleq 6\theta - 2$.

Here again, replacing this expression into (6) does not provide a computable expression, even if the additional hypothesis that the signal $\{X_t, t \geq 0\}$ is a diffusion process is introduced.

4 Piecewise constant approximation of the signal process

The purpose of this section is to investigate the effect of replacing the signal process $\{X_t, t \geq 0\}$ by a piecewise constant process $\{\overline{X}_t, t \geq 0\}$ whose values on "pieces" are related in some way to the values taken by the original signal process at some particular instants. This is formalized in the following

Definition 4.1 *A process $\{\overline{X}_t, t \geq 0\}$ is subordinate to the process $\{X_t, t \geq 0\}$ relatively to the partition π if, for all $i \geq 0$*

$$
\overline{X}_t \text{ is } \mathcal{F}_{t_{i+1}}\text{-measurable}, \; t_i \leq t \leq t_{i+1} .
$$

The following example provide a particular class of subordinate process, to be used throughout the paper.

Example. For all $i \geq 0$ are given

- a partition $\{A_i^j, 1 \leq j \leq k(i)\}$ of the time interval $[t_i, t_{i+1})$,
- an increasing sequence

$$
t_i \leq \tau_i^1 < \cdots < \tau_i^j < \cdots < \tau_i^{k(i)} \leq t_{i+1} .
$$

Then the piecewise constant process $\{\overline{X}_t, t \geq 0\}$ defined by

$$
\overline{X}_t = X_{\tau_i^j} , \quad \text{if } t \in A_i^j .
$$

is subordinate to $\{X_t, t \geq 0\}$ relatively to the partition π. There is a similar class of subordinate processes, where the time interval to be partitioned is rather $(t_i, t_{i+1}]$.

The problem is to chose $\{\overline{X}_t, t \geq 0\}$ in such a way that the conditional law of $\overline{X}_{t_i}$ given $\overline{\mathcal{Y}}_{t_i}$ is more simple to handle than the conditional law of X_{t_i} given $\overline{\mathcal{Y}}_{t_i}$, and is even computable in the particular case where the signal $\{X_t, t \geq 0\}$ is a diffusion process.

Introduce

$$
\overline{Z}_t^s \triangleq \exp\left\{ \int_s^t h^*(\overline{X}_\tau)\, dY_\tau - \tfrac{1}{2} \int_s^t |h(\overline{X}_\tau)|^2\, d\tau \right\} \tag{9}
$$

and $\overline{Z}_t \triangleq \overline{Z}_t^0$. Under the reference probability measure $P^\dagger$, the processes $\{\overline{X}_t, t \geq 0\}$ and $\{Y_t, t \geq 0\}$ are independent, so that the stochastic integral in (9) is well defined, although $\{\overline{X}_t, t \geq 0\}$ is not necessarily adapted. Therefore, it is possible for all $T > 0$ to define a new probability measure $\overline{P}$ equivalent on $[0,T]$ to $P^\dagger$ with Radon-Nikodym derivative $\overline{Z}_T$, so that under $\overline{P}$

$$
dY_t = h(\overline{X}_t)\, dt + d\overline{V}_t ,
$$

where $\{\overline{V}_t, t \geq 0\}$ is a standard Wiener process, independent of $\{\overline{X}_t, t \geq 0\}$.

The problem is now to estimate $\overline{X}_{t_i}$ from $\overline{\mathcal{Y}}_{t_i}$, i.e. to compute the conditional law of $\overline{X}_{t_i}$ given $\overline{\mathcal{Y}}_{t_i}$. By the Bayes formula

$$
\overline{\mathbf{E}}(\phi(\overline{X}_{t_i}) \mid \overline{\mathcal{Y}}_{t_i}) = \frac{\mathbf{E}^\dagger(\phi(\overline{X}_{t_i}) \overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_i})}{\mathbf{E}^\dagger(\overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_i})} ,
$$

so that it is enough to compute $\{\overline{p}_i\,,\, i \geq 0\}$ defined by

$$(\overline{p}_i, \phi) \triangleq \mathbf{E}^\dagger(\phi(\overline{X}_{t_i})\overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_i}) \; .$$

It follows from the proof of Proposition 3.2 that

$$(\overline{p}_{i+1}, \phi) = \mathbf{E}^\dagger([\overline{U}_{i+1}\phi]\overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_{i+1}}) \; , \qquad (10)$$

where, for all $i \geq 0$

$$\overline{\Xi}^{t_i}_{t_{i+1}} \triangleq \mathbf{E}^\dagger(\overline{Z}^{t_i}_{t_{i+1}} \mid \mathcal{F}_{t_{i+1}} \vee \overline{\mathcal{Y}}^{t_i}_{t_{i+1}}) \; ,$$

$$\overline{U}_{i+1}\phi \triangleq \mathbf{E}^\dagger(\phi(\overline{X}_{t_{i+1}})\overline{\Xi}^{t_i}_{t_{i+1}} \mid \mathcal{F}_{t_i} \vee \overline{\mathcal{Y}}^{t_i}_{t_{i+1}}) \; .$$

Going back to the examples introduced above, some particular piecewise constant subordinate processes will be considered, the corresponding expression for $\overline{\Xi}^{t_i}_{t_{i+1}}$ will be derived, and it will be checked whether or not the additional hypothesis that the signal $\{X_t\,,\, t \geq 0\}$ is a diffusion process can lead to computable expressions.

Example 1 (Continued). For the sampling procedure defined by ξ_i, $\overline{\Xi}^{t_i}_{t_{i+1}}$ has the same form than (7) where now

$$h_i \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} h(\overline{X}_s)\, ds \; .$$

Two different piecewise constant subordinate processes will be considered.

$\boxed{\mathbf{1a}}$ Define

$$\overline{X}_t = X_{t_i} \; , \qquad \text{if } t_i \leq t < t_{i+1} \; .$$

Then $h_i = h(X_{t_i})$ and $\overline{\Xi}^{t_i}_{t_{i+1}} = \Psi_i(X_{t_i})$, where for all $x \in \mathbf{R}^m$

$$\Psi_i(x) \triangleq \exp\left\{h^*(x)\xi_i - \tfrac{1}{2}|h(x)|^2\delta_i\right\} \; . \qquad (11)$$

Therefore

$$\overline{U}_{i+1}\phi = \Psi_i(X_{t_i})\, \mathbf{E}^\dagger(\phi(X_{t_{i+1}}) \mid \mathcal{F}_{t_i}) \; .$$

Under the additional hypothesis that the signal $\{X_t\,,\, t \geq 0\}$ is a diffusion process with semi-group $\{P_t\,,\, t \geq 0\}$

$$\overline{U}_{i+1}\phi = \Psi_i(X_{t_i})\, [P_{\delta_i}\phi](X_{t_i}) \; ,$$

and

$$\begin{aligned}(\overline{p}_{i+1}, \phi) &= \mathbf{E}^\dagger(\Psi_i(X_{t_i})\, [P_{\delta_i}\phi](X_{t_i})\overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_{i+1}}) \\ &= (\overline{p}_i, \Psi_i[P_{\delta_i}\phi]) \; ,\end{aligned}$$

so that $\{\overline{p}_i\,,\, i \geq 0\}$ satisfies the following recurrence

$$\overline{p}_{i+1} = P^*_{\delta_i}[\Psi_i \overline{p}_i] \; , \qquad (12)$$

which is a computable expression, and can be considered as a time discretization scheme for the Zakai equation (3). The rate of convergence of this approximation will be considered in Section 5.

$\boxed{\mathbf{1b}}$ Define

$$\overline{X}_t = X_{t_{i+1}} \; , \qquad \text{if } t_i < t \leq t_{i+1} \; .$$

Then $h_i = h(X_{t_{i+1}})$ and $\overline{\Xi}^{t_i}_{t_{i+1}} = \Psi_i(X_{t_{i+1}})$. Therefore

$$\overline{U}_{i+1}\phi = \mathbf{E}^\dagger(\phi(X_{t_{i+1}})\, \Psi_i(X_{t_{i+1}}) \mid \mathcal{F}_{t_i} \vee \overline{\mathcal{Y}}^{t_i}_{t_{i+1}}) \; .$$

Under the additional hypothesis that the signal $\{X_t\,,\, t \geq 0\}$ is a diffusion process with semi-group $\{P_t\,,\, t \geq 0\}$

$$\overline{U}_{i+1}\phi = P_{\delta_i}[\Psi_i\phi](X_{t_i}) \; ,$$

and

$$\begin{aligned}(\overline{p}_{i+1}, \phi) &= \mathbf{E}^\dagger(P_{\delta_i}[\Psi_i\phi](X_{t_i})\overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_{i+1}}) \\ &= (\overline{p}_i, P_{\delta_i}[\Psi_i\phi]) \; .\end{aligned}$$

This results in the following recurrence

$$\overline{p}_{i+1} = \Psi_i[P^*_{\delta_i}\overline{p}_i] \; , \qquad (13)$$

which gives another time discretization scheme for the Zakai equation (3).

Remark 4.2 In the numerical scheme (12) (resp. (13)) the transition from $\overline{p}_i$ to $\overline{p}_{i+1}$ reflects the following situation: A new measurement ξ_i is available, which is a compression of the information provided by $\{z_t\,,\, t_i \leq t \leq t_{i+1}\}$ according to (4). This measurement is interpreted as a noisy nonlinear observation of X_{t_i} (resp. $X_{t_{i+1}}$), and is combined with the current estimate $\overline{p}_i$ of X_{t_i} to produce an estimate $\overline{p}_{i+1}$ of $X_{t_{i+1}}$.

Example 2 (Continued). For the sampling procedure defined by $(\xi^\sharp_i, \xi^\flat_i)$, $\overline{\Xi}^{t_i}_{t_{i+1}}$ has the same form than (8) where now

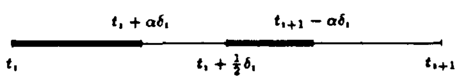
$$h^\sharp_i \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \varpi\Big(\frac{s - t_i}{t_{i+1} - t_i}\Big)\, h(\overline{X}_s)\, ds \; .$$

$$h^\flat_i \triangleq \frac{1}{\delta_i} \int_{t_i}^{t_{i+1}} \varpi\Big(\frac{t_{i+1} - s}{t_{i+1} - t_i}\Big)\, h(\overline{X}_s)\, ds \; .$$

The following family, parametrized by $0 < \alpha < \frac{1}{2}$, of piecewise constant subordinate processes will be considered

$$\overline{X}_t \triangleq \begin{cases} X_{t_i} & \text{if } t \in A^\alpha_i \\ X_{t_{i+1}} & \text{if } t \in [t_i, t_{i+1}) \setminus A^\alpha_i \end{cases}$$

where for all $i \geq 0$, A^α_i denotes the following subset of the time interval $[t_i, t_{i+1})$



It is then possible to find a particular value α_0 for which

$$h^\sharp_i = h(X_{t_{i+1}}) \; , \qquad h^\flat_i = h(X_{t_i}) \; .$$

Therefore (8) becomes

$$\begin{aligned}\overline{\Xi}^{t_i}_{t_{i+1}} &= \Psi^\sharp_i(X_{t_{i+1}})\, \Psi^\flat_i(X_{t_i}) \\ &\quad \cdot \exp\left\{\tfrac{1}{12}|h(X_{t_{i+1}}) - h(X_{t_i})|^2\delta_i\right\}\end{aligned}$$

where for all $x \in \mathbf{R}^m$

$$\Psi_i^\sharp(x) \triangleq \exp\left\{h^*(x)\xi_i^\sharp - \tfrac{1}{4}|h(x)|^2\delta_i\right\} ,$$

$$\Psi_i^\flat(x) \triangleq \exp\left\{h^*(x)\xi_i^\flat - \tfrac{1}{4}|h(x)|^2\delta_i\right\} ,$$

and

$$\overline{U}_{i+1}\phi = \Psi_i^\flat(X_{t_i})\, \mathbf{E}^\dagger(\phi(X_{t_{i+1}})\, \Psi_i^\sharp(X_{t_{i+1}}) \cdot \exp\left\{\tfrac{1}{12}|h(X_{t_{i+1}}) - h(X_{t_i})|^2\delta_i\right\} \mid \mathcal{F}_{t_i} \vee \overline{\mathcal{Y}}_{t_{i+1}}^{t_i}) .$$

Under the additional hypothesis that the signal $\{X_t\, , t \geq 0\}$ is a diffusion process with semi-group $\{P_t\, , t \geq 0\}$

$$\overline{U}_{i+1}\phi = \Psi_i^\flat(X_{t_i})\, Q_{\delta_i}[\Psi_i^\sharp\phi](X_{t_i}) ,$$

and

$$\begin{aligned}(\overline{p}_{i+1}, \phi) &= \mathbf{E}^\dagger(\Psi_i^\flat(X_{t_i})\, Q_{\delta_i}[\Psi_i^\sharp\phi](X_{t_i})\overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_{i+1}}) \\ &= (\overline{p}_i, \Psi_i^\flat Q_{\delta_i}[\Psi_i^\sharp\phi]) ,\end{aligned}$$

so that $\{\overline{p}_i\, , i \geq 0\}$ satisfies the following recurrence

$$\overline{p}_{i+1} = \Psi_i^\sharp Q_{\delta_i}^*[\Psi_i^\flat\overline{p}_i] , \tag{14}$$

where the family of operators $\{Q_\delta\, , \delta \geq 0\}$ is defined by

$$Q_\delta\phi \triangleq \mathbf{E}^\dagger(\phi(X_{t+\delta})\exp\left\{\tfrac{1}{12}|h(X_{t+\delta}) - h(X_t)|^2\delta\right\} \mid \mathcal{F}_t) .$$

Note that $\Psi_i^\sharp(x)\ \Psi_i^\flat(x) = \Psi_i(x)$, and that the operator Q_{δ_i} can be seen as a perturbation of the semi-group P_{δ_i}. However, it is not obvious that (14) is a computable expression and can be considered as a time discretization of the Zakai equation (3). The relevant analysis and the rate of convergence of this approximation will be considered elsewhere.

5 A product formula and its rate of convergence

The purpose of this section is to study, from the point of view of numerical analysis, the following recurrence

$$\overline{p}_{i+1} = P_{\delta_i}^*[\Psi_i\overline{p}_i] , \tag{15}$$

derived in the previous section, as a time discretization scheme for the Zakai equation

$$dp_t = L^* p_t\, dt + h^* p_t\, dY_t . \tag{16}$$

Recall that

$$(p_{t_i}, \phi) = \mathbf{E}^\dagger(\phi(X_{t_i})Z_{t_i} \mid \mathcal{Y}_{t_i}) ,$$

$$(\overline{p}_i, \phi) = \mathbf{E}^\dagger(\phi(\overline{X}_{t_i})\overline{Z}_{t_i} \mid \overline{\mathcal{Y}}_{t_i}) ,$$

so that $\overline{p}_i$ should be "close" to p_{t_i}. Indeed it will be proved below that

$$\left\{\mathbf{E}^\dagger|\overline{p}_i - p_{t_i}|^2\right\}^{1/2} \leq C\delta ,$$

where δ is the mesh of the partition π up to time t_i, and $|\cdot|$ denotes the norm in the Sobolev space $L^2(\mathbf{R}^m)$.

Remark 5.1 Similar rate of convergence has already been obtained for approximation of nonlinear filtering problems, in Picard [6] and Newton [4]. The proof in [6] uses only probabilistic arguments and does not consider the Zakai equation, but rather the underlying nonlinear filtering problem. In [4], the Zakai equation is considered for pure–jump Markov processes rather than diffusion processes, and the approximation procedure relies on the stochastic Taylor formula of Wagner–Platen [7,8].

Define, for all $x \in \mathbf{R}^m$

$$\Psi_t^s(x) \triangleq \exp\left\{h^*(x)\,(Y_t - Y_s) - \tfrac{1}{2}|h(x)|^2\,(t-s)\right\} .$$

Note that two operators are involved in (16)

- the unbounded operator L^* which generates the adjoint semi-group $\{P_t^*\, , t \geq 0\}$,
- the multiplication operator B which generates the two-parameter stochastic semi-group $\{\Psi_t^s\, , 0 \leq s \leq t\}$,

so that the time discretization scheme (15) is a Trotter-like product formula for the Zakai equation (16). See Bensoussan–Glowinski–Rascanu [1] for a related work in this direction.

The main assumption of this section is that the signal $\{X_t\, , t \geq 0\}$ is a diffusion process

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t\, , \qquad X_0 \sim p_0(x)\, dx$$

observed in continuous time through measurements

$$dY_t = h(X_t)\, dt + dV_t .$$

Define $a \triangleq \sigma\sigma^*$ and $\overline{a}^i \triangleq \sum_{j=1}^m \frac{\partial a^{i,j}}{\partial x_j}$. The coefficients satisfy the following hypotheses

(i) p_0 *is a density on* $\mathbf{R}^m$,

(ii) σ *is a continuous and bounded function on* $\mathbf{R}^m$ *and* a *is a uniformly elliptic* $m \times m$ *matrix, i.e.* $a(x) \geq \alpha I$,

(iii) b *and* $\overline{a}$ *are bounded and measurable functions from* $\mathbf{R}^m$ *to* $\mathbf{R}^m$,

(iv) h *is a measurable and bounded function from* $\mathbf{R}^m$ *to* $\mathbf{R}^d$.

The infinitesimal generator of the semi-group $\{P_t\, , t \geq 0\}$ is defined by

$$L \triangleq \tfrac{1}{2}\sum_{i,j=1}^m a^{i,j}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b^i \frac{\partial}{\partial x_i} ,$$

and satisfies, under the hypotheses, the following coercivity property: for all $u \in H^1(\mathbf{R}^m)$

$$2\langle Lu, u\rangle + \mu\|u\|^2 \leq \lambda|u|^2 , \tag{17}$$

where $\|\cdot\|$ denotes the norm in the Sobolev space $H^1(\mathbf{R}^m)$. Existence and uniqueness of a solution to equation (16) is proved in Pardoux [5] and Krylov–Rozovskii [3].

Theorem 5.2 *Suppose that, in addition to (i)-(iv)*

(v) a, b and $\bar{a}$ have bounded first derivative,

(vi) h has bounded derivatives up to order 2 .

Then, if $p_0 \in H^1(\mathbf{R}^m)$

$$\max_{0 \le k \le i} \left\{ \mathbf{E}^\dagger |p_{t_k} - \bar{p}_k|^2 \right\}^{1/2} \le C\delta . \qquad (18)$$

PROOF. Under the hypotheses, it follows from Theorem 2.1 of [5] that $p \in L^2(\Omega; C([0,T]; H^1(\mathbf{R}^m)))$. Also, for all $i \ge 0$, $\bar{p}_i \in L^2(\Omega; H^1(\mathbf{R}^m))$ and in addition

$$\max_{0 \le k \le i} \mathbf{E}^\dagger \|\bar{p}_k\|^2 \le C .$$

For $t \ge t_k$, define $v_t \triangleq P^*_{t-t_k}[\Psi^{t_k}_t \bar{p}_k]$, so that $\bar{p}_k = v_{t_k}$ and $\bar{p}_{k+1} = v_{t_{k+1}}$. Differentiating with resp ~t to t gives

$$\begin{aligned} dv_t &= L^* v_t \, dt + \left\{ P^*_{t-t_k}[B \Psi^{t_k}_t \bar{p}_k] \right\}^* dY_t \\ &= L^* v_t \, dt + [Bp_t]^* \, dY_t + \beta^*_t \, dY_t , \end{aligned}$$

where the perturbation term is defined by

$$\beta_t \triangleq [P^*_{t-t_k} B - B P^*_{t-t_k}] [\Psi^{t_k}_t \bar{p}_k] .$$

Note that $\beta \in L^2(\Omega; C([t_k, T]; H^1(\mathbf{R}^m)))$. The identity of energy of [5] applied to the difference $\epsilon = v - p$, and the coercivity property (17) give

$$\mathbf{E}^\dagger |\epsilon_t|^2 \le \mathbf{E}^\dagger |\epsilon_{t_k}|^2 + C \int_{t_k}^t \mathbf{E}^\dagger |\epsilon_s|^2 \, ds + C' \int_{t_k}^t \mathbf{E}^\dagger |\beta_s|^2 \, ds .$$

Assume the following estimate holds

$$\mathbf{E}^\dagger |\beta_s|^2 \le C |s - t_k|^2 \, \exp\{C(s - t_k)\} \, \mathbf{E}^\dagger \|\bar{p}_k\|^2 . \quad (19)$$

Applying Gronwall's lemma and setting $t = t_{k+1}$, yields

$$\begin{aligned} &\mathbf{E}^\dagger |\bar{p}_{k+1} - p_{t_{k+1}}|^2 \\ &\le [\mathbf{E}^\dagger |\bar{p}_k - p_{t_k}|^2 + C |t_{k+1} - t_k|^3] \, \exp\{C(t_{k+1} - t_k)\} , \end{aligned}$$

and the rate of convergence (18) follows from the discrete Gronwall lemma. The end of the proof is devoted to proving estimate (19).

First, the following perturbation result holds

$$\begin{aligned} &[P^*_{t-t_k} B - B P^*_{t-t_k}] \, u \\ &= \int_{t_k}^t P^*_{t-s} \, [L^* B - B L^*] \, P^*_{s-t_k} u \, ds , \end{aligned}$$

provided u is smooth enough. Under the hypotheses, $[L^* B - B L^*]$ is a bounded operator from $H^1(\mathbf{R}^m)$ to $L^2(\mathbf{R}^m)$, so that it is enough that $u \in H^1(\mathbf{R}^m)$ for (20) to hold. Now, $[\Psi^{t_k}_t \bar{p}_k] \in H^1(\mathbf{R}^m)$ a.s. so that

$$\beta_t = \int_{t_k}^t P^*_{t-s} \, [L^* B - B L^*] \, P^*_{s-t_k} \, [\Psi^{t_k}_t \bar{p}_k] \, ds .$$

Therefore

$$|\beta_t|^2 \le C |t - t_k|^2 \, \|\Psi^{t_k}_t \bar{p}_k\|^2 ,$$

and

$$\begin{aligned} \mathbf{E}^\dagger |\beta_t|^2 &\le C |t - t_k|^2 \, \mathbf{E}^\dagger \|\Psi^{t_k}_t \bar{p}_k\|^2 \\ &\le C |t - t_k|^2 \, \exp\{C(t - t_k)\} \, \mathbf{E}^\dagger \|\bar{p}_k\|^2 , \end{aligned}$$

which proves (19). □

Remark 5.3 The same rate of convergence holds for the approximation scheme (13).

The next step is to approximate the adjoint semigroup $\{P^*_t , t \ge 0\}$ itself, i.e. to approximate the associated Fokker-Planck equation. For instance, using an implicit Euler scheme results in the following approximation scheme

$$(I - \delta_i L^*) \bar{p}_{i+1} = \Psi_i \bar{p}_i .$$

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TIME-DISCRETIZATION OF THE ZAKAI EQUATION FOR DIFFUSION PROCESSES OBSERVED IN CORRELATED NOISE*

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A time discretization scheme is provided for the Zakai equation, a stochastic PDE which gives the conditional density of a diffusion process observed in white-noise. The case where the observation noise and the state noise are correlated, is considered. The numerical scheme is based on a Trotter-like product formula, which exhibits *prediction* and *correction* steps, and for which an error estimate of order δ is proved, where δ is the time discretization step. The *correction* step is associated with a degenerate second-order stochastic PDE, for which a representation result in terms of stochastic characteristics has been proved by Krylov–Rozovskii [13] and Kunita [15, 17]. A discretization scheme is then provided to approximate these stochastic characteristics. Under the additional assumption that the correlation coefficient is constant, an error estimate of order $\sqrt{\delta}$ is proved for the overall numerical scheme. This has been proved to be the best possible error estimate by Elliott–Glowinski [7].

KEY WORDS: Diffusion processes, correlated noises, nonlinear filtering, Zakai equation, stochastic PDE, stochastic characteristics, time discretization.

1. INTRODUCTION

The purpose of this paper is to present a *computable* time discretization scheme for the Zakai equation of nonlinear filtering with correlated noises, and to provide an estimate of the rate of convergence.

In the case of independent noises, the problem has been studied by Kushner [18], Newton [21], Korezlioglu–Mazziotto [11], Bennaton [1], DiMasi–Pratelli–Runggaldier [6], Picard [22], Bensoussan–Glowinski–Rascanu [2] and Le Gland [20]. Some of these authors have actually considered the associated Zakai equation. Time discretization schemes have been provided with a rate of convergence of order δ, where δ is the time discretization step.

In the case of correlated noises, the problem has been studied by Elliott–Glowinski [7]. The best approximation of the continuous filter based on the

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values of the observation process at a regular partition (with mesh δ) has been considered, and it has been proved that the rate of convergence is of order $\sqrt{\delta}$. However, no algorithm is provided to actually *compute* this approximation.

The paper is organized as follows. In Section 2, the nonlinear filtering problem is presented. Some results on the Zakai equation, and on a related degenerate second-order stochastic PDE, are recalled in Section 3. A Trotter-like product formula is then considered, with an error estimate of order δ. However, this numerical scheme is not *computable*. In Section 4, a representation result in terms of *stochastic characteristics* is presented for the degenerate second-order stochastic PDE. This part follows mainly the work of Krylov–Rozovskii [13]—see also Kunita [15, 17]. A time discretization scheme is presented in Section 5, based on an approximation of the stochastic characteristics. Under the additional assumption that the correlation coefficient is constant, an error estimate of order $\sqrt{\delta}$ can be proved. In addition, this numerical scheme is actually *computable*, as far as time discretization is concerned, i.e. up to space discretization.

2. THE FILTERING PROBLEM

Consider the following stochastic differential system, defined on the probability space $(\Omega, \mathscr{F}, P)$

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t + \rho(X_t)\,dV_t$$

$$dY_t = h(X_t)\,dt + dV_t$$

where the non observed component $\{X_t, t \geqq 0\}$ takes values in $\mathbf{R}^m$, and the observation $\{Y_t, t \geqq 0\}$ takes values in $\mathbf{R}^d$. $\{W_t, t \geqq 0\}$ and $\{V_t, t \geqq 0\}$ are independent Wiener processes of appropriate dimension, with covariance matrix I (identity) and r respectively. For the clarity of exposition, it is assumed throughout the paper that $r = I$. In addition, the random variable X_0 is independent of the Wiener processes, with probability distribution $p_0(x)\,dx$.

Throughout the paper, it is assumed that the coefficients, b, σ, ρ and h are globally Lipschitz continuous functions defined on $\mathbf{R}^m$, so that the stochastic differential system has a unique strong solution. The following definitions are used: $a \triangleq \sigma\sigma^*$ and $c \triangleq \rho\rho^*$. In particular, it is not assumed that either a or c is uniformly elliptic.

With the diffusion process $\{X_t, t \geqq 0\}$ are associated the two partial differential operators

$$L \triangleq \frac{1}{2} \sum_{i,j=1}^{m} [a^{i,j} + c^{i,j}] \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b^i \frac{\partial}{\partial x_i},$$

$$L_0 \triangleq \frac{1}{2} \sum_{i,j=1}^{m} a^{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b^i \frac{\partial}{\partial x_i}.$$

Another family of partial differential operators to be considered is

$$B_k \triangleq h_k + \sum_{i=1}^{m} \rho_k^i \frac{\partial}{\partial x_i}, \quad 1 \leqq k \leqq d.$$

Introducing

$$Z_t^s \triangleq \exp\left\{\int_s^t h^*(X_\tau)\, dY_\tau - \frac{1}{2}\int_s^t |h(X_\tau)|^2\, d\tau\right\}, \quad Z_t \triangleq Z_t^0,$$

it is standard that, for all $T > 0$ the original probability measure P is equivalent on $[0, T]$ to the *reference probability* measure $P\dagger$ with Radon–Nikodym derivative Z_T, so that under $P\dagger$

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t + \rho(X_t)[dY_t - h(X_t)\, dt], \tag{2.1}$$

where $\{W_t, t \geqq 0\}$ and $\{Y_t, t \geqq 0\}$ are independent Wiener processes, with covariance matrix I (identity), and the random variable X_0 is independent of the Wiener processes, with probability distribution $p_0(x)\, dx$.

The Bayes formula gives

$$\mathbf{E}(f(X_t) \mid \mathscr{Y}_t) = \frac{\mathbf{E}\dagger(f(X_t)Z_t \mid \mathscr{Y}_t)}{\mathbf{E}\dagger(Z_t \mid \mathscr{Y}_t)},$$

and in addition

$$\mathbf{E}\dagger(f(X_t)Z_t \mid \mathscr{Y}_t) = \int f(x) p_t(x)\, dx,$$

where the unnormalized conditional density $\{p_t, t \geqq 0\}$ satisfies the Zakai equation [25]

$$dp_t = L^* p_t\, dt + \sum_{k=1}^{d} B_k^* p_t\, dY_t^k. \tag{2.2}$$

Consider then the following decomposition of the Zakai equation (2.2)

$$dp_t = L_0^* p_t\, dt + \Lambda^* p_t\, dt + \sum_{k=1}^{d} B_k^* p_t\, dY_t^k,$$

where

$$\Lambda \triangleq L - L_0 = \frac{1}{2} \sum_{i,j=1}^{m} c^{i,j} \frac{\partial^2}{\partial x_i \partial x_j}.$$

On one hand, the partial differential operator L_0 generates a strongly continuous semigroup $\{P_t, t \geqq 0\}$. On the other hand, it is possible to associate a stochastic semigroup $\{Q_t^s, 0 \leqq s \leqq t\}$ with the following degenerate second-order stochastic PDE

$$dp_t = \Lambda^* p_t \, dt + \sum_{k=1}^{d} B_k^* p_t \, dY_t^k, \tag{2.3}$$

which is studied below. Therefore, it is worth studying the following Trotter-like product formulas for approximating the original Zakai equation (2.2)

$$\begin{aligned} \bar{p}_{i+1} &= P_{\delta_i}^* Q_{t_{i+1}}^{t_i} \bar{p}_i, \\ \bar{p}_{i+1} &= Q_{t_{i+1}}^{t_i} P_{\delta_i}^* \bar{p}_i, \end{aligned} \tag{2.4}$$

where $\delta_i \triangleq t_{i+1} - t_i$, and $0 = t_0 < t_1 < \cdots < t_i < \cdots$
The main interest of such product formulas is that the original equation has been split into a second-order deterministic PDE (*prediction* step), and a degenerate second-order stochastic PDE (*correction* step). In the case of independent noises, this stochastic PDE reduces to a zero-order equation, for which there exists a straightforward explicit solution. In the case of correlated noises, a representation result is available by the method of stochastic characteristics (i.e. involving the stochastic flow of diffeomorphism associated with a SDE driven by the observation process), see Krylov–Rozovskii [13] and Section 4 below.

Remark 2.1 A similar prediction-correction numerical scheme was obtained by Kushner [18] in the case of independent noises.

Remark 2.2 Written in Stratonovich form, equation (2.3) is a first-order stochastic PDE. For such an equation, one can use the representation result of Kunita [15, 17], and translate the stochastic characteristics equations from Stratonovich form back to Itô form, to recover the representation result of [13].

As a consequence of the above discussion, there will be two steps in designing the approximation to the original Zakai equation (2.2)

- first use a Trotter-like product formula,
- then approximate the solution of the degenerate second-order stochastic PDE, by approximating the stochastic flow of diffeomorphisms involved in the stochastic characteristics method of [13].

It will be proved that the first step can be achieved with a rate of convergence of order δ, whereas the rate of convergence for the second step (and *a fortiori* for the global approximation procedure) is of order $\sqrt{\delta}$ only, where $\delta \triangleq \max_{i \geqq 0} \delta_i$.

3. TROTTER-LIKE PRODUCT FORMULA

For all $n \geqq 0$, let H^n denote the space of real-valued Lebesgue-measurable functions

on $\mathbf{R}^m$ whose generalized derivatives up to order n are square-integrable, with norm $\|\cdot\|_n$

$$\|u\|_n^2 \triangleq \sum_{0\leq|\alpha|\leq n} \int |D^\alpha u(x)|^2\,dx < \infty.$$

In addition, the following shorthand notations will be used throughout the paper: $|\cdot| \triangleq \|\cdot\|_0$ and $\|\cdot\| \triangleq \|\cdot\|_1$.

The beginning of this section is devoted to recall existence, uniqueness and regularity results for the Zakai equation

$$dp_t = L^* p_t\,dt + \sum_{k=1}^{d} B_k^* p_t\,dY_t^k, \tag{3.1}$$

and the degenerate second-order stochastic PDE

$$dp_t = \Lambda^* p_t\,dt + \sum_{k=1}^{d} B_k^* p_t\,dY_t^k, \tag{3.2}$$

with semigroup $\{Q_t^s, 0\leq s\leq t\}$.

Although no coercivity hypothesis is satisfied, the following result is proved in Krylov–Rozovskii [13].

THEOREM 3.1 *Let $n\geq 1$ be fixed. Assume that*

- *a and c have bounded derivatives up to order* $\max(n,2)$,
- *b, ρ and h have bounded derivatives up to order n,*
- *the initial condition satisfies $p_0 \in H^n$.*

Then both Eqs. (3.1) and (3.2) have a unique solution $p\in M^2(0,T;H^n)$. In addition

$$p\in L^2(\Omega; C_w([0,T]; H^n)),$$

and the following estimate holds

$$\mathbf{E}^\dagger\left[\sup_{0\leq t\leq T} \|p_t\|_n^2\right] \leq \|p_0\|_n^2\, e^{CT}.$$

Similarly, for the Fokker–Planck equation

$$p_t' = L_0^* p_t, \tag{3.3}$$

and the following deterministic PDE associated with (3.2)

$$p_t' = \Lambda^* p_t, \tag{3.4}$$

with semigroup $\{P_t^*, t \geqq 0\}$ and $\{T_t^*, t \geqq 0\}$ respectively, it holds

THEOREM 3.2 *Let $n \geqq 1$ be fixed. Assume that*

- *a and c have bounded derivatives up to order* $\max(n, 2)$,
- *b has bounded derivatives up to order n,*
- *the initial condition satisfies $p_0 \in H^n$.*

Then both Eqs. (3.3) and (3.4) have a unique solution $p \in L^2(0, T; H^n)$. In addition

$$p \in C_w([0, T]; H^n),$$

and the following estimate holds

$$\sup_{0 \leqq t \leqq T} \|p_t\|_n^2 \leqq \|p_0\|_n^2 \, e^{CT}.$$

Remark 3.3 In the case where the coefficients a and c are uniformly elliptic, a slightly stronger theorem holds, see Krylov–Rozovskii [12] and Pardoux [23].

□ *Error Estimate*

The purpose here is to study one of the Trotter-like product formulas (2.4).

THEOREM 3.4 *Consider the following approximation scheme*

$$\bar{p}_{i+1} = P_{\delta_i}^* Q_{t_{i+1}}^{t_i} \bar{p}_i. \tag{3.5}$$

Assume that

- *a c, b, ρ, and h have bounded derivatives up to order* 3,
- *the initial condition satisfies $p_0 \in H^3$.*

Then $\bar{p}_i$ approximates the solution p_{t_i} of the original Zakai equation (3.1) with a rate of convergence of order δ. Indeed

$$\{\mathbf{E}^\dagger |\bar{p}_i - p_{t_i}|^2\}^{1/2} \leqq C\delta \|p_0\|_3.$$

Proof The idea is to get an equation for $v_t \triangleq P_{t-s}^* Q_t^s \phi$ with ϕ smooth enough, that is similar to the original Zakai equation for p_t, except for some perturbation terms which have to be estimated. This gives an estimate of the one-step error, and the global estimate is obtained using the Gronwall lemma.

Differentiating with respect to t

$$dv_t = L_0^* v_t \, dt + P_{t-s}^* \left[\Lambda^* Q_t^s \phi \, dt + \sum_{k=1}^{d} B_k^* Q_t^s \phi \, dY_t^k \right]$$

$$= L_0^* v_t\, dt + \Lambda^* v_t\, dt + \sum_{k=1}^{d} B_k^* v_t\, dY_t^k$$

$$+ [P_{t-s}^* \Lambda^* - \Lambda^* P_{t-s}^*] Q_t^s \phi\, dt + \sum_{k=1}^{d} [P_{t-s}^* B_k^* - B_k^* P_{t-s}^*] Q_t^s \phi\, dY_t^k$$

$$=: L^* v_t\, dt + \sum_{k=1}^{d} B_k^* v_t\, dY_t^k + f_t\, dt + \sum_{k=1}^{d} g_t^k\, dY_t^k,$$

where the perturbation terms are defined by

$$f_t \triangleq [P_{t-s}^* \Lambda^* - \Lambda^* P_{t-s}^*] Q_t^s \phi \quad \text{and} \quad g_t^k \triangleq [P_{t-s}^* B_k^* - B_k^* P_{t-s}^*] Q_t^s \phi,$$

respectively. The difference $\varepsilon_t \triangleq v_t - p_t$ satisfies

$$d\varepsilon_t = L^* \varepsilon_t\, dt + \sum_{k=1}^{d} B_k^* \varepsilon_t\, dY_t^k + f_t\, dt + \sum_{k=1}^{d} g_t^k\, dY_t^k.$$

Using estimates of [13]

$$\mathbf{E}^\dagger |\varepsilon_t|^2 \leqq \left[\mathbf{E}^\dagger |\varepsilon_s|^2 + C\mathbf{E}^\dagger \int_s^t |f_\tau|^2\, d\tau + C\mathbf{E}^\dagger \sum_{k=1}^{d} \int_s^t \|g_\tau^k\|^2\, d\tau \right] e^{C(t-s)}.$$

Assume that the following estimates hold

$$\mathbf{E}^\dagger |f_\tau|^2 \leqq C(\tau - s)^2 \mathbf{E}^\dagger \|\phi\|_3^2\, e^{C(\tau - s)}, \tag{3.6}$$

$$\mathbf{E}^\dagger \|g_\tau^k\|^2 \leqq C(\tau - s)^2 \mathbf{E}^\dagger \|\phi\|_3^2\, e^{C(\tau - s)}. \tag{3.7}$$

Then the Gronwall lemma would yield

$$\mathbf{E}^\dagger |\varepsilon_t|^2 \leqq [\mathbf{E}^\dagger |\varepsilon_s|^2 + C(t-s)^3 \mathbf{E}^\dagger \|\phi\|_3^2]\, e^{C(t-s)},$$

provided $\phi \in L^2(\Omega; H^3)$. Now, it follows from the assumptions and from Theorem 3.1, that $\bar{p}_i \in L^2(\Omega; H^3)$ for all i, so that setting $s = t_i$, $t = t_{i+1}$ and $\phi = \bar{p}_i$

$$\mathbf{E}^\dagger |\bar{p}_{i+1} - p_{t_{i+1}}|^2 \leqq [\mathbf{E}^\dagger |\bar{p}_i - p_{t_i}|^2 + C(t_{i+1} - t_i)^3 \mathbf{E}^\dagger \|\bar{p}_i\|_3^2]\, e^{C(t_{i+1} - t_i)},$$

and the result follows from the discrete Gronwall lemma. The end of the proof is devoted to proving estimates (3.6) and (3.7).

□ *Estimate (3.6)*

The following perturbation result

$$[P^*_{\tau-s}\Lambda^* - \Lambda^* P^*_{\tau-s}]u = \int_s^\tau P^*_{\tau-\tau'}[L^*_0\Lambda^* - \Lambda^* L^*_0]P^*_{\tau'-s}u\, d\tau',$$

holds for u smooth enough. It follows from the assumptions, that the partial differential operator $D \triangleq [L^*_0\Lambda^* - \Lambda^* L^*_0]$ is bounded from H^3 to H^0. In addition, it follows from Theorem 3.2 that $\{P^*_t, t \geqq 0\}$ is a strongly continuous semigroup in both H^0 and H^3. Therefore

$$|f_\tau| \leqq \int_s^\tau |P^*_{\tau-\tau'} D P^*_{\tau'-s} Q^s_t \phi|\, d\tau' \leqq C(\tau-s)\|Q^s_t\phi\|_3\, e^{C(\tau-s)}.$$

Then

$$\mathbf{E}^\dagger|f_\tau|^2 \leqq C(\tau-s)^2\mathbf{E}^\dagger\|Q^s_t\phi\|_3^2\, e^{C(\tau-s)} \leqq C(\tau-s)^2\mathbf{E}^\dagger_i\|\phi\|_3^2\, e^{C(\tau-s)}.$$

□ *Estimate (3.7)*

Similarly, the following perturbation result

$$[P^*_{\tau-s}B^*_k - B^*_k P^*_{\tau-s}]u = \int_s^\tau P^*_{\tau-\tau'}[L^*_0 B^*_k - B^*_k L^*_0]P^*_{\tau'-s}u\, d\tau',$$

holds for u smooth enough. It follows from the assumptions, that the partial differential operator $D_k \triangleq [L^*_0 B^*_k - B^*_k L^*_0]$ is bounded from H^3 to H^1. In addition, it follows from Theorem 3.2 that $\{P^*_t, t \geqq 0\}$ is a strongly continuous semigroup in both H^1 and H^3. Therefore

$$\|g^k_\tau\| \leqq \int_s^\tau \|P^*_{\tau-\tau'} D_k P^*_{\tau'-s} Q^s_t\phi\| d\tau' \leqq C(\tau-s)\|Q^s_t\phi\|_3\, e^{C(\tau-s)}.$$

Then

$$\mathbf{E}^\dagger\|g^k_\tau\|^2 \leqq C(\tau-s)^2\mathbf{E}^\dagger\|Q^s_t\phi\|_3^2\, e^{C(\tau-s)} \leqq C(\tau-s)^2\mathbf{E}^\dagger\|\phi\|_3^2\, e^{C(\tau-s)}. \qquad \square$$

Remark 3.5 In the case where the coefficient a is uniformly elliptic, the same error estimate can be proved under weaker regularity assumptions on the coefficients and the initial condition, see Florchinger-LeGland [8].

Remark 3.6 It is possible to approximate the stochastic differential equation (2.1), in such a way that the approximation $\bar{p}_i$ given by (2.4), is actually the conditional density of the approximate process at time t_i, given the observations $\mathcal{Y}_{t_i}$. This problem will be addressed elsewhere.

The approximation scheme (3.5) is not yet *computable*. First, the Fokker–Planck equation (3.3) with semigroup $\{P^*_t, t \geqq 0\}$, has to be approximated: this is a rather standard problem, for which one can use e.g. the backward Euler scheme, or some other approximation scheme. On the other hand, some representation results are

presented in the next section, which can be used for the approximation of the degenerate second-order stochastic PDE (3.2), with semigroup $\{Q_t^s,\ 0 \leq s \leq t\}$.

4. STOCHASTIC CHARACTERISTICS

Parallel to the decomposition of the stochastic PDE (2.2), there is a similar decomposition for the stochastic differential equation (2.1). With the first component

$$dX_t = b(X_t)\,dt + \overset{\bullet}{\sigma}(X_t)\,dW_t,$$

is associated the partial differential operator L_0 and the Fokker–Planck equation (3.3). It is proved below that the second component

$$dX_t = \rho(X_t)[dY_t - h(X_t)\,dt], \tag{4.1}$$

is associated with the degenerate second-order stochastic PDE (3.2) and the corresponding deterministic PDE (3.4).

The beginning of this section is devoted to recall results concerning the stochastic flow of diffeomorphisms associated with the stochastic differential equation (4.1).

THEOREM 4.1 *Let $\xi_{s,t}(\cdot)$ be the stochastic flow associated with the forward stochastic differential equation*

$$d\xi_t = \rho(\xi_t)[dY_t - h(\xi_t)\,dt]. \tag{4.2}$$

Assume that the coefficients h and ρ have bounded derivatives up to order $(n+1)$. Then $\xi_{s,t}(\cdot)$ is a C^n-diffeomorphism in $\mathbf{R}^m$.

Under the assumption that the coefficient ρ has bounded derivatives up to order 2, the inverse map $\xi_{s,t}^{-1}(\cdot)$ is given explicitly as the (backward) stochastic flow $\eta_{t,s}(\cdot)$ associated with the backward stochastic differential equation

$$d\eta_t = \rho(\eta_t) \oplus [dY_t - h(\eta_t)\,dt] - \rho_0(\eta_t)\,dt, \tag{4.3}$$

with

$$\rho_0^i \triangleq \sum_{k=1}^{d} \sum_{j=1}^{m} \frac{\partial \rho_k^i}{\partial x_j} \rho_k^j, \quad 1 \leq i \leq m.$$

The regularity of $\xi_{s,t}(\cdot)$ was first proved by Blagoveschenskii–Freidlin [3], whereas the rest of the theorem is proved in Kunita [16].

PROPOSITION 4.2 *The Jacobian $J_{s,t}(\cdot)$ (i.e. the determinant of the Jacobian matrix) of the diffeomorphism $\xi_{s,t}(\cdot)$ satisfies*

$$J_{s,t}(x) \triangleq \exp\left\{\int_s^t \alpha^*(\xi_{s,\tau}(x))[dY_\tau - h(\xi_{s,\tau}(x))\,d\tau]\right.$$

$$\left. - \int_s^t h_0(\xi_{s,\tau}(x))\,d\tau - \int_s^t \bar{\alpha}(\xi_{s,\tau}(x))\,d\tau\right\} \tag{4.4}$$

with

$$\alpha_k \triangleq \sum_{i=1}^m \frac{\partial \rho_k^i}{\partial x_i} = \operatorname{div} \rho_k, \quad 1 \leqq k \leqq d$$

$$\bar{\alpha} \triangleq \frac{1}{2} \sum_{k=1}^d \sum_{i,j=1}^m \frac{\partial \rho_k^i}{\partial x_j} \frac{\partial \rho_k^j}{\partial x_i} \quad \textit{and} \quad h_0 \triangleq \sum_{k=1}^d \sum_{i=1}^m \frac{\partial h_k}{\partial x_i} \rho_k^i.$$

Proof Transform first the stochastic differential equation (4.2) into Stratonovich form

$$d\xi_t = \rho(\xi_t) \circ [dY_t - h(\xi_t)\,dt] - \tfrac{1}{2}\rho_0(\xi_t)\,dt.$$

Similarly to the Liouville formula for ordinary differential equations, see Hartman [10], it holds

$$d \log J_{s,t}(x) = \alpha^*(\xi_{s,t}(x)) \circ [dY_t - h(\xi_{s,t}(x))\,dt] - h_0(\xi_{s,t}(x))\,dt - \tfrac{1}{2} \operatorname{div} \rho_0(\xi_{s,t}(x))\,dt.$$

Transforming back to Itô form

$$d \log J_{s,t}(x) = \alpha^*(\xi_{s,t}(x))[dY_t - h(\xi_{s,t}(x))\,dt] - h_0(\xi_{s,t}(x))\,dt$$

$$- \tfrac{1}{2} \operatorname{div} \rho_0(\xi_{s,t}(x))\,dt + \tfrac{1}{2}\alpha_0(\xi_{s,t}(x))\,dt.$$

Now it holds

$$\operatorname{div} \rho_0 = \sum_{k=1}^d \sum_{i,j=1}^m \frac{\partial}{\partial x_i}\left(\frac{\partial \rho_k^i}{\partial x_j} \rho_k^j\right),$$

$$\alpha_0 \triangleq \sum_{k=1}^d \sum_{i=1}^m \frac{\partial \alpha_k}{\partial x_i} \rho_k^i = \sum_{k=1}^d \sum_{i,j=1}^m \frac{\partial^2 \rho_k^j}{\partial x_i \partial x_j} \rho_k^i,$$

which finishes the proof. □

Remark 4.3 Note that $[J_{s,t}(\eta_{t,s}(\cdot))]^{-1}$ is actually the Jacobian of the inverse diffeomorphism $\eta_{t,s}(\cdot)$.

Define

$$\Xi_{s,t}(x) \triangleq \exp\left\{\int_s^t h^*(\xi_{s,\tau}(x))\, dY_\tau - \frac{1}{2}\int_s^t |h(\xi_{s,\tau}(x))|^2\, d\tau\right\}, \tag{4.5}$$

and

$$\Theta_{s,t}(x) \triangleq \Xi_{s,t}(x)[J_{s,t}(x)]^{-1} = \exp\left\{\int_s^t h^*(\xi_{s,\tau}(x))\, dY_\tau - \frac{1}{2}\int_s^t |h(\xi_{s,\tau}(x))|^2\, d\tau\right.$$

$$\left. - \int_s^t \alpha^*(\xi_{s,\tau}(x))[dY_\tau - h(\xi_{s,\tau}(x))\, d\tau] + \int_s^t h_0(\xi_{s,\tau}(x))\, d\tau + \int_s^t \bar{\alpha}(\xi_{s,\tau}(x))\, d\tau\right\}.$$

Introduce the following *definition*

$$Q_t^s q(x) \triangleq q(\eta_{t,s}(x))\Theta_{s,t}(\eta_{t,s}(x)), \tag{4.6}$$

or equivalently

$$Q_t^s q(\xi_{s,t}(x)) = q(x)\Theta_{s,t}(x).$$

where the same notation has been used as in the previous section. This will be justified by the Theorem 4.8 to be proved below.

Remark 4.4 Under the additional assumption that the coefficient ρ has bounded derivatives up to order 2, the Lemma 6.2 of [16, Chapter 2] gives the following explicit expressions in terms of backward Itô stochastic integrals

$$\Xi_{s,t}(\eta_{t,s}(x)) = \exp\left\{\int_s^t h^*(\eta_{t,\tau}(x)) \oplus dY_\tau - \frac{1}{2}\int_s^t |h(\eta_{t,\tau}(x))|^2\, d\tau - \int_s^t h_0(\eta_{t,\tau}(x))\, d\tau\right\},$$

$$J_{s,t}(\eta_{t,s}(x)) = \exp\left\{\int_s^t \alpha^*(\eta_{t,\tau}(x)) \oplus [dY_\tau - h(\eta_{t,\tau}(x))\, d\tau]\right.$$

$$\left. - \int_s^t h_0(\eta_{t,\tau}(x))\, d\tau - \int_s^t \bar{\alpha}(\eta_{t,\tau}(x))\, d\tau - \int_s^t \alpha_0(\eta_{t,\tau}(x))\, d\tau\right\},$$

where the coefficients h_0 and α_0 have already been defined as

$$h_0 \triangleq \sum_{k=1}^d \sum_{i=1}^m \frac{\partial h_k}{\partial x_i}\rho_k^i \quad \text{and} \quad \alpha_0 \triangleq \sum_{k=1}^d \sum_{i=1}^m \frac{\partial \alpha_k}{\partial x_i}\rho_k^i = \sum_{k=1}^d \sum_{i,j=1}^m \frac{\partial^2 \rho_k^j}{\partial x_i \partial x_j}\rho_k^i.$$

Therefore

$$\Gamma_{t,s}(x) \triangleq \Theta_{s,t}(\eta_{t,s}(x)) = \exp\left\{\int_s^t h^*(\eta_{t,\tau}(x)) \oplus dY_\tau - \frac{1}{2}\int_s^t |h(\eta_{t,\tau}(x))|^2 \, d\tau \right.$$

$$\left. - \int_s^t \alpha^*(\eta_{t,\tau}(x)) \oplus [dY_\tau - h(\eta_{t,\tau}(x))\, d\tau] + \int_s^t \bar{\alpha}(\eta_{t,\tau}(x))\, d\tau + \int_s^t \alpha_0(\eta_{t,\tau}(x))\, d\tau \right\}. \quad (4.7)$$

Remark 4.5 If $\rho \equiv 0$, then $\xi_{s,t}(x) = x$ so that

$$Q_t^s q(x) = q(x) \exp\{h^*(x)(Y_t - Y_s) - \tfrac{1}{2}|h(x)|^2(t-s)\},$$

which is actually the explicit solution of the equation

$$dq_t = \sum_{k=1}^{d} h_k q_t \, dY_t^k,$$

with initial condition q at time s. In this case, (2.4) reduces to the discretization schemes considered in [2, 18, 20].

First, the following *stability* result holds

PROPOSITION 4.6 *Let $n \geqq 0$ be fixed. Assume that*

- *c, ρ and h have bounded derivatives up to order $(n+1)$,*
- *the initial condition satisfies $q \in H^n$.*

Then $Q_t^s q$ is a square integrable random variable with values in H^n. In addition, the following estimate holds

$$\{\mathbf{E}^\dagger \|Q_t^s q\|_n^2\}^{1/2} \leqq \|q\|_n \, e^{C(t-s)}.$$

Proof It is enough to prove the result for $n = 0$.

Using the change of variable $x = \eta_{t,s}(y)$ i.e. $y = \xi_{s,t}(x)$

$$\mathbf{E}^\dagger |Q_t^s q|^2 = \mathbf{E}^\dagger \int [|q(\eta_{t,s}(y))| \Theta_{s,t}(\eta_{t,s}(y))]^2 \, dy = \int |q(x)|^2 \mathbf{E}\{\Theta_{s,t}(x)\} \, dx,$$

and the result follows from the estimate

$$\sup_{x \in \mathbf{R}^m} \mathbf{E}\{\Theta_{s,t}(x)\} \leqq e^{C(t-s)}. \qquad \square$$

Another property of the two-parameter stochastic semigroup $\{Q_t^s, 0 \leqq s \leqq t\}$ is provided by the following

PROPOSITION 4.7 *Let $\{T_t, t \geqq 0\}$ be the semigroup generated by*

$$\Lambda = \frac{1}{2} \sum_{i,j=1}^{m} c^{i,j} \frac{\partial^2}{\partial x_i \partial x_j}.$$

Then

$$\mathbf{E}\dagger Q_t^s = T_{t-s}^*.$$

Proof Using the same change of variable as in the proof of Proposition 4.5, it holds

$$(\mathbf{E}\dagger(Q_t^s q), f) = \mathbf{E}\dagger \int q(\eta_{t,s}(y))\Theta_{s,t}(\eta_{t,s}(y)) f(y)\, dy = \int q(x)\mathbf{E}[f(\xi_{s,t}(x))]\, dx,$$

for any test-function f. Now, under the original probability measure P

$$d\xi_t = \rho(\xi_t)\, dV_t,$$

where $\{V_t, t \geqq 0\}$ is a Wiener process with covariance matrix I. Therefore

$$(\mathbf{E}\dagger(Q_t^s q), f) = (q, T_{t-s} f) = (T_{t-s}^* q, f). \qquad \square$$

The following representation result of the solution of Eq. (3.2) in terms of the *stochastic characteristics* $\eta_{t,s}(\cdot)$, is the stochastic counterpart of the usual method of characteristics for linear first-order PDE. It has been proved by Krylov–Rozovskii [13] and Kunita [15, 17].

THEOREM 4.8 *Let $\{Q_t^s, s \leqq t\}$ be defined by (4.6). Then, the unique solution of equation (3.2) satisfies*

$$q_t(x) = Q_t^s q_s(x). \tag{4.8}$$

Proof The proof given below is essentially that of [13]. Introduce

$$\zeta_t^s \triangleq \exp\left\{\int_s^t \phi_\tau^* \, dY_\tau - \frac{1}{2}\int_s^t |\phi_\tau|^2 \, d\tau\right\},$$

where $\{\phi_\tau, s \leqq \tau \leqq t\}$ is deterministic.

It follows from the Itô formula that $\bar{q}_\tau \triangleq \mathbf{E}\dagger(\zeta_\tau^s \cdot q_\tau)$ satisfies

$$\bar{q}_\tau' = \Lambda^* \bar{q}_\tau + \sum_{k=1}^d B_k^* \bar{q}_\tau \phi_\tau^k, \tag{4.9}$$

with the initial condition $\bar{q}_s = \mathbf{E}\dagger(q_s)$. On the other hand, define

$$\bar{w}_\tau(x) \triangleq \mathbf{E}^\phi\left[f(\xi_{\tau,t}(x)) \exp\left\{\int_\tau^t \phi_{\tau'}^* h(\xi_{\tau,\tau'}(x))\, d\tau'\right\}\right],$$

where under the probability P^ϕ

$$d\xi_\tau = \rho(\xi_\tau)[dV_\tau^\phi + \phi_\tau\, d\tau],$$

and $\{V_t^\phi, t \geqq 0\}$ is a Wiener process with covariance matrix I. By the Feynman–Kac formula, $\{\bar{w}_\tau, s \leqq \tau \leqq t\}$ satisfies a PDE which is dual to (4.9), so that $(\bar{q}_t, f) = (\bar{q}_s, \bar{w}_s)$. Consider now the right-hand side in the representation result (4.8). Then

$$\zeta_t^s \cdot Q_t^s q_s(x) = q_s(\eta_{t,s}(x)) \Xi_{s,t}^\phi(\eta_{t,s}(x)) \exp\left\{\int_s^t \phi_\tau^* h(\eta_{t,\tau}(x))\, d\tau\right\} [J_{s,t}(\eta_{t,s}(x))]^{-1},$$

with

$$\Xi_{s,t}^\phi(x) \triangleq \exp\left\{\int_s^t [h(\xi_{s,\tau}(x)) + \phi_\tau]^* \, dY_\tau - \frac{1}{2}\int_s^t |h(\xi_{s,\tau}(x)) + \phi_\tau|^2\, d\tau\right\}.$$

Define next $\bar{v}_t \triangleq \mathbf{E}^\dagger(\zeta_t^s \cdot Q_t^s q_s)$. The Fubini theorem, the change of variable $x = \eta_{t,s}(y)$ and the Lemma 6.2 of [16, Chapter 2] give

$$\begin{aligned}
(\bar{v}_t, f) &= \mathbf{E}^\dagger \int f(y) q_s(\eta_{t,s}(y)) \Xi_{s,t}^\phi(\eta_{t,s}(y)) \exp\left\{\int_s^t \phi_\tau^* h(\eta_{t,\tau}(y))\, d\tau\right\} [J_{s,t}(\eta_{t,s}(y))]^{-1}\, dy \\
&= \mathbf{E}^\dagger \int f(\xi_{s,t}(x)) q_s(x) \Xi_{s,t}^\phi(x) \exp\left\{\int_s^t \phi_\tau^* h(\xi_{s,\tau}(x))\, d\tau\right\} dx \\
&= \int \bar{q}_s(x) \mathbf{E}^\dagger\left[f(\xi_{s,t}(x)) \Xi_{s,t}^\phi(x) \exp\left\{\int_s^t \phi_\tau^* h(\xi_{s,\tau}(x))\, d\tau\right\}\right] dx \\
&= \int \bar{q}_s(x) \mathbf{E}^\phi\left[f(\xi_{s,t}(x)) \exp\left\{\int_s^t \phi_\tau^* h(\xi_{s,\tau}(x))\, d\tau\right\}\right] dx = (\bar{q}_s, \bar{w}_s).
\end{aligned}$$

It follows that $(\bar{q}_t, f) = (\bar{v}_t, f)$ for arbitrary test-function f and arbitrary $\{\phi_\tau, s \leqq \tau \leqq t\}$, which finishes the proof. □

5. APPROXIMATION OF THE STOCHASTIC CHARACTERISTICS

It has been proved in Section 4 that the stochastic semigroup $\{Q_t^s,\ 0 \leqq s \leqq t\}$ associated with the degenerate second-order stochastic PDE (3.2) satisfies

$$Q_t^s \phi(x) = \phi(\eta_{t,s}(x)) \Gamma_{t,s}(x), \tag{5.1}$$

where $\eta_{t,s}(\cdot)$ is the inverse of the stochastic flow of diffeomorphisms $\xi_{s,t}(\cdot)$ associated with the stochastic differential equation (4.2), and $\Gamma_{t,s}(x)$ has been

defined in (4.7). The purpose of this section is to investigate approximations of (5.1).

Considering that $\eta_{t,s}(\cdot)$ is also the stochastic flow of diffeomorphisms associated with the backward stochastic differential equation (4.3), it is natural to consider the following approximation

$$\bar{Q}_t^s\phi(x) \triangleq \phi(\bar{\eta}_{t,s}(x))\bar{\Gamma}_{t,s}(x), \tag{5.2}$$

where

$$\bar{\eta}_{t,s}(x) \triangleq x - \rho(x)[Y_t - Y_s - h(x)(t-s)] + \rho_0(x)(t-s),$$

and

$$\bar{\Gamma}_{t,s}(x) \triangleq \exp\{h^*(x)(Y_t - Y_s) - \tfrac{1}{2}|h(x)|^2(t-s) - \alpha^*(x)[Y_t - Y_s - h(x)(t-s)]$$
$$+ \bar{\alpha}(x)(t-s) + \alpha_0(x)(t-s)\},$$

are *computable* approximations of $\eta_{t,s}(x)$ and $\Gamma_{t,s}(x)$ respectively, both depending only on the increments $(Y_t - Y_s)$.

Remark 5.1 One possible approach would be to approximate $\eta_{t,s}(\cdot)$ by the stochastic flow of diffeomorphisms associated with the ordinary differential equation obtained from (4.3) by replacing the observation sample-path $\{Y_t, 0 \leqq t \leqq T\}$ with some regular approximation, such as the Euler stepwise approximation or the polygonal interpolation. The numerical analysis of such an approximation should not be very difficult. However, the resulting approximation would not be *explicitly computable.*

The remainder of this section is devoted to studying the rate of convergence of this approximation. First, a *stability* result similar to Proposition 4.6 is needed.

Condition A *Let $n \geqq 0$ be fixed. Assume that the initial condition satisfies $q \in H^n$.*

Then $\bar{Q}_t^s q$ is a square integrable random variable with values in H^n. In addition, the following estimate holds

$$\{\mathbf{E}^\dagger \|\bar{Q}_t^s q\|_n^2\}^{1/2} \leqq \|q\|_n e^{C(t-s)}.$$

Remark 5.2 Because $\bar{\eta}_{t,s}(\cdot)$ is *not* a diffeomorphism, this stability result can not be proved in the same way as in the proof of Proposition 4.6. The following proposition, which is proved in the Appendix, shows that Condition (A) holds in the simple case where the correlation coefficient ρ is constant. Whether this remains true in the general case—or how to modify the approximation scheme in such a way that Condition (A) holds without any additional assumption on the correlation coefficient—is still an open problem (however, see Remark A.1 below).

Proposition 5.3 *Let $n \geqq 0$ be fixed. Assume that*

- *ρ is constant,*
- *h has bounded derivatives up to order n.*

Then Condition (A) holds.

Remark 5.4 The approximations $\bar{\eta}_{t,s}(x)$ and $\bar{\Gamma}_{t,s}(x)$ are based on the explicit expressions for $\eta_{t,s}(x)$ and $\Gamma_{t,s}(x)$, given in (4.3) and (4.7) respectively. This explains why the regularity assumptions on the coefficient h are different in Proposition 4.6 and Proposition 5.3.

Remark 5.5 In the case where the correlation coefficient ρ is constant, the approximations of $\eta_{t,s}(x)$ and $\Gamma_{t,s}(x)$ take the simple form

$$\bar{\eta}_{t,s}(x) \triangleq x - \rho[Y_t - Y_s - h(x)(t-s)],$$

and

$$\bar{\Gamma}_{t,s}(x) \triangleq \exp\{h^*(x)(Y_t - Y_s) - \tfrac{1}{2}|h(x)|^2(t-s)\},$$

respectively.

Next, the following proposition provides an error estimate for commuting the operator $\bar{Q}^s_t$ and spatial derivatives.

PROPOSITION 5.6 *Let $n \geqq 0$ and α a multi-index, be fixed. Assume that*

- *ρ has bounded derivatives up to order $(n+|\alpha|+2)$,*
- *h has bounded derivatives up to order $(n+|\alpha|)$,*
- *the initial condition satisfies $q \in H^{n+|\alpha|}$.*

Then, under Condition (A)

$$\{\mathbf{E}^\dagger \|\bar{Q}^s_t D^\alpha q - D^\alpha \bar{Q}^s_t q\|_n^2\}^{1/2} \leqq C\sqrt{t-s}\,\|q\|_{n+|\alpha|}.$$

Here again, the proof of this proposition is given in the Appendix.

□ *Overall Error Estimate*

The main result of the paper is provided by the following

THEOREM 5.7 *Consider the following approximation scheme*

$$\bar{p}_{i+1} = P^*_{\delta_i} \bar{Q}^{t_i}_{t_{i+1}} \bar{p}_i.$$

Assume that

- *a and c have bounded derivatives up to order 4,*
- *b and ρ have bounded derivatives up to order 3,*
- *h has bounded derivatives up to order 2,*
- *the initial condition satisfies $p_0 \in H^2$.*

Then, under Condition (A), $\bar{p}_i$ approximates the solution p_{t_i} of the original equation (3.1) with a rate of convergence of order $\sqrt{\delta}$. Indeed

$$\{\mathbf{E}^\dagger|\bar{p}_i - p_{t_i}|^2\}^{1/2} \leqq C\sqrt{\delta}\|p_0\|_2.$$

Proof In view of Theorem 3.4, it is enough to prove that

$$\{\mathbf{E}^\dagger|\bar{p}_i - \bar{p}_i|^2\}^{1/2} \leqq C\sqrt{\delta}\|p_0\|_2.$$

Similarly to the proof of Theorem 3.4, the idea is to get an equation for $\bar{Q}_t^s\phi$ with ϕ smooth enough, that is similar to the original Eq. (3.2) for $Q_t^s\psi$, except for the initial condition and for some perturbation terms which have to be estimated. This gives an estimate of the one-step error, and the global estimate is obtained using the Gronwall lemma. Throughout the proof, the summation convention over repeated indices i, j, is used.

Differentiating both sides of (5.2) with respect to t

$$\begin{aligned}
d\bar{Q}_t^s\phi(x) &= \frac{\partial\phi}{\partial x_i}(\bar{\eta}_{t,s}(x))\left[-\sum_{k=1}^{d}\rho_k^i(x)[dY_t^k - h_k(x)\,dt] + \rho_0^i(x)\,dt\right]\bar{\Gamma}_{t,s}(x) \\
&\quad + \frac{1}{2}\frac{\partial^2\phi}{\partial x_i\,\partial x_j}(\bar{\eta}_{t,s}(x))\sum_{k=1}^{d}[\rho_k^i(x)\rho_k^j(x)\,dt]\bar{\Gamma}_{t,s}(x) \\
&\quad + \phi(\bar{\eta}_{t,s}(x))\left[\sum_{k=1}^{d}h_k(x)\,dY_t^k - \frac{1}{2}|h(x)|^2\,dt - \sum_{k=1}^{d}\alpha_k(x)[dY_t^k - h_k(x)\,dt]\right. \\
&\quad \left. + \bar{\alpha}(x)\,dt + \alpha_0(x)\,dt + \frac{1}{2}|h(x) - \alpha(x)|^2\,dt\right]\bar{\Gamma}_{t,s}(x) \\
&\quad + \frac{\partial\phi}{\partial x_i}(\bar{\eta}_{t,s}(x))\left[-\sum_{k=1}^{d}\rho_k^i(x)[h_k(x) - \alpha_k(x)]\,dt\right]\bar{\Gamma}_{t,s}(x) \\
&= \frac{1}{2}c^{i,j}(x)\frac{\partial^2\phi}{\partial x_i\,\partial x_j}(\bar{\eta}_{t,s}(x))\bar{\Gamma}_{t,s}(x)\,dt \\
&\quad + \left[\rho_0^i(x) + \sum_{k=1}^{d}\rho_k^i(x)\alpha_k(x)\right]\frac{\partial\phi}{\partial x_i}(\bar{\eta}_{t,s}(x))\bar{\Gamma}_{t,s}(x)\,dt \\
&\quad + [\bar{\alpha}(x) + \alpha_0(x) + \tfrac{1}{2}|\alpha(x)|^2]\phi(\bar{\eta}_{t,s}(x))\bar{\Gamma}_{t,s}(x)\,dt \\
&\quad + \sum_{k=1}^{d}[h_k(x) - \alpha_k(x)]\phi(\bar{\eta}_{t,s}(x))\bar{\Gamma}_{t,s}(x)\,dY_t^k
\end{aligned}$$

$$-\sum_{k=1}^{d} \rho_k^i(x) \frac{\partial \phi}{\partial x_i}(\bar{\eta}_{t,s}(x)) \Gamma_{t,s}(x)\, dY_t^k.$$

Now, it can be checked that

$$\bar{\alpha} + \alpha_0 + \frac{1}{2}|\alpha|^2 = \frac{1}{2} \frac{\partial^2 c^{i,j}}{\partial x_i \partial x_j} \quad \text{and} \quad \rho_0^i + \sum_{k=1}^{d} \rho_k^i \alpha_k = \frac{\partial c^{i,j}}{\partial x_j}.$$

Therefore, it holds

$$\begin{aligned} d\bar{Q}_t^s \phi(x) = & \left[\frac{1}{2} c^{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j}(\bar{\eta}_{t,s}(x)) \right. \\ & \left. + \frac{\partial c^{i,j}}{\partial x_j}(x) \frac{\partial \phi}{\partial x_i}(\bar{\eta}_{t,s}(x)) + \frac{1}{2} \frac{\partial^2 c^{i,j}}{\partial x_i \partial x_j}(x) \phi(\bar{\eta}_{t,s}(x)) \right] \Gamma_{t,s}(x)\, dt \\ & + \sum_{k=1}^{d} \left[h_k(x) \phi(\bar{\eta}_{t,s}(x)) - \rho_k^i(x) \frac{\partial \phi}{\partial x_i}(\bar{\eta}_{t,s}(x)) - \frac{\partial \rho_k^i}{\partial x_i}(x) \phi(\bar{\eta}_{t,s}(x)) \right] \Gamma_{t,s}(x)\, dY_t^k \\ = & \left[\frac{1}{2} c^{i,j}(x) \bar{Q}_t^s \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x) + \frac{\partial c^{i,j}}{\partial x_j}(x) \bar{Q}_t^s \frac{\partial \phi}{\partial x_i}(x) + \frac{1}{2} \frac{\partial^2 c^{i,j}}{\partial x_i \partial x_j}(x) \bar{Q}_t^s \phi(x) \right] dt \\ & + \sum_{k=1}^{d} \left[h_k(x) \bar{Q}_t^s \phi(x) - \rho_k^i(x) \bar{Q}_t^s \frac{\partial \phi}{\partial x_i}(x) - \frac{\partial \rho_k^i}{\partial x_i}(x) \bar{Q}_t^s \phi(x) \right] dY_t^k \end{aligned}$$

so that

$$\begin{aligned} d\bar{Q}_t^s \phi = & \Lambda^* \bar{Q}_t^s \phi\, dt + \sum_{k=1}^{d} B_k^* \bar{Q}_t^s \phi\, dY_t^k \\ & + \frac{1}{2} c^{i,j} \left[\bar{Q}_t^s \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_i \partial x_j} \bar{Q}_t^s \phi \right] dt + \frac{\partial c^{i,j}}{\partial x_j} \left[\bar{Q}_t^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \bar{Q}_t^s \phi \right] dt \\ & - \sum_{k=1}^{d} \rho_k^i \left[\bar{Q}_t^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \bar{Q}_t^s \phi \right] dY_t^k. \end{aligned}$$

The difference $\varepsilon_t \triangleq \bar{Q}_t^s \phi - Q_t^s \psi$ satisfies

$$d\varepsilon_t = \Lambda^* \varepsilon_t\, dt + \sum_{k=1}^{d} B_k^* \varepsilon_t\, dY_t^k + f_t\, dt + \sum_{k=1}^{d} g_t^k\, dY_t^k,$$

where the perturbation terms are defined by

$$f_\tau \triangleq \frac{1}{2} c^{i,j} \left[\bar{Q}_\tau^s \frac{\partial^2 \phi}{\partial x_i \partial x_j} - \frac{\partial^2}{\partial x_i \partial x_j} \bar{Q}_\tau^s \phi \right] + \frac{\partial c^{i,j}}{\partial x_j} \left[\bar{Q}_\tau^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \bar{Q}_\tau^s \phi \right],$$

and

$$g_\tau^k \triangleq -\rho_k^i \left[\bar{Q}_\tau^s \frac{\partial \phi}{\partial x_i} - \frac{\partial}{\partial x_i} \bar{Q}_\tau^s \phi \right],$$

respectively. Using estimates of [13]

$$\mathbf{E}^\dagger |\varepsilon_t|^2 \leqq \left[\mathbf{E}^\dagger |\varepsilon_s|^2 + C \mathbf{E}^\dagger \int_s^t |f_\tau|^2 \, d\tau + C \mathbf{E}^\dagger \sum_{k=1}^d \int_s^t \|g_\tau^k\|^2 \, d\tau \right] e^{C(t-s)}.$$

Moreover, it follows from Proposition 5.6 that

$$\mathbf{E}^\dagger |f_\tau|^2 \leqq C(\tau - s) \mathbf{E}^\dagger \|\phi\|_2^2 \, e^{C(\tau - s)},$$

$$\mathbf{E}^\dagger \|g_\tau^k\|^2 \leqq C(\tau - s) \mathbf{E}^\dagger \|\phi\|_2^2 \, e^{C(\tau - s)},$$

and therefore the Gronwall lemma yields

$$\mathbf{E}^\dagger |\varepsilon_t|^2 \leqq [\mathbf{E}^\dagger |\varepsilon_s|^2 + C(t-s)^2 \mathbf{E}^\dagger \|\phi\|_2^2] \, e^{C(t-s)},$$

provided $\phi \in H^2$. Now, it follows from the assumptions and in particular Condition (A), that $\bar{p}_i \in L^2(\Omega; H^2)$ for all i, so that setting $s = t_i$, $t = t_{i+1}$, $\phi = \bar{p}_i$ and $\psi = \bar{p}_i$

$$\mathbf{E}^\dagger |\bar{Q}_{t_{i+1}}^{t_i} \bar{p}_i - Q_{t_{i+1}}^{t_i} \bar{p}_i|^2 \leqq [\mathbf{E}^\dagger |\bar{p}_i - \bar{p}_i|^2 + C(t_{i+1} - t_i)^2 \mathbf{E}^\dagger \|\bar{p}_i\|_2^2] \, e^{C(t_{i+1} - t_i)}.$$

Next

$$\begin{aligned} \mathbf{E}^\dagger |\bar{p}_{i+1} - \bar{p}_{i+1}|^2 &= \mathbf{E}^\dagger |P_{\delta_i}^* [\bar{Q}_{t_{i+1}}^{t_i} \bar{p}_i - Q_{t_{i+1}}^{t_i} \bar{p}_i]|^2 \\ &\leqq [\mathbf{E}^\dagger |\bar{p}_i - \bar{p}_i|^2 + C(t_{i+1} - t_i)^2 \mathbf{E}^\dagger \|\bar{p}_i\|_2^2] \, e^{C(t_{i+1} - t_i)}, \end{aligned}$$

and the result follows from the discrete Gronwall lemma. □

A further step in the time-discretization would consist in approximating the Fokker–Planck semigroup $\{P_t^*, t \geqq 0\}$, using some classical approximation scheme. For instance, using the backward Euler scheme would result in the following global approximation scheme

$$(I - \delta_i L_0^*) \bar{p}_{i+1} = \bar{Q}_{t_{i+1}}^{t_i} \bar{p}_i,$$

with the same error estimate.

□ *Particle Approximation*

Another possible approach to approximate the degenerate second-order stochastic PDE (3.2)—based also on the representation (5.1) in terms of stochastic characteristics—would be to use *particle methods*, adapting the results presented in Raviart [24] for deterministic first-order PDE. The basic idea is to solve exactly Eq. (3.2) for an approximation of the initial condition, rather than approximate the stochastic characteristics as was done before.

Suppose that, at time t_i an approximation of the conditional probability distribution $q(x)\,dx$ is available, in terms of a convex linear combination of Dirac masses sitting at some particle locations $\{x_i^k, k \in K\}$ with corresponding weights $\{a_i^k, k \in K\}$ i.e.

$$q(x)\,dx \sim \sum_{k \in K} a_i^k \delta(x - x_i^k). \tag{5.3}$$

Solving exactly Eq. (3.2) in weak sense, with the approximation (5.3) as initial condition, gives the following approximation

$$Q_{t_{i+1}}^{t_i} q(x)\,dx \sim \sum_{k \in K} a_{i+1}^k \delta(x - x_{i+1}^k)$$

for the solution at time t_{i+1}. The new particle locations $\{x_{i+1}^k, k \in K\}$ and the corresponding weights $\{a_{i+1}^k, k \in K\}$ are computed according to

$$x_{i+1}^k = \xi_{t_i, t_{i+1}}(x_i^k) \quad \text{and} \quad a_{i+1}^k = a_i^k \Xi_{t_i, t_{i+1}}(x_i^k),$$

where $\xi_{s,t}(\cdot)$ is the diffeomorphism associated with Eq. (4.2), and $\Xi_{s,t}(\cdot)$ has been defined in (4.5).

The error estimate associated with this particle approximation will be studied elsewhere.

6. CONCLUSION

A time-discretization scheme of the Zakai equation for diffusion processes observed in correlated noise has been proposed, based on the stochastic characteristics introduced in [13, 15, 17]. Under the additional assumption that the correlation coefficient is constant, it has been shown that the rate of convergence of this approximation is of order $\sqrt{\delta}$, where δ is the time discretization step.

The same rate of convergence has been obtained in Elliott–Glowinski [7] for a different approximation

- on one hand, the approximation considered in [7] has a probabilistic interpretation, which is not the case so far for the time discretization scheme presented here (however, see Remark 3.6 above),

- on the other hand, the latter is *actually computable*, whereas no numerical algorithm is provide to *compute* the approximation considered in [7].

Another point of interest would be to study some particle approximation for the degenerate second-order stochastic PDE, adapting the results presented in Raviart [24] for deterministic first-order PDE.

As was pointed out to the authors by Harold Kushner and the anonymous referee, one would have to discretize the space variable and to bound the state space, in order to get a completely computable numerical scheme. This is a different problem, for which several approaches have already been used: finite difference approximation, by Kushner [18] and DiMasi–Runggaldier [5], finite element method, by Bennaton [1] and Germani–Piccioni [9], with error estimate. The reference [9] also provides error estimate for bounding the state space, using weighted Sobolev spaces introduced by Krylov–Rozovskii [14]. Therefore, the time discretization scheme presented in the paper should be combined with such space discretization techniques, in order to be completely computable. To some extent, the choice of the space discretization scheme is dependent on the application: for instance, the method of characteristics (also called particle approximation in [24]) is well-adapted to first-order PDE arising in the filtering of noise-free processes, and has been recently used in target tracking applications, see Campillo–Le Gland [4] and Lasdas–Davis [19].

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APPENDIX

Proof of Stability and Commutation Estimates

The purpose of this appendix is to prove the stability and commutation estimates for the approximation introduced in Section 5.

Proof of Proposition 5.3 It is enough to prove the result for $n=0$.

Since $\bar{\eta}_{t,s}(\cdot)$ is not a diffeomorphism, one can not use a change of variable as in the proof of Proposition 4.6. Instead, one uses the fact that $\bar{\eta}_{t,s}(x)$ and $\Gamma_{t,s}(x)$ are very simple functions of the Gaussian random variable $(Y_t - Y_s)$. First

$$\mathbf{E}^{\dagger}|\bar{Q}_t^s q|^2 = \mathbf{E}^{\dagger} \int [|q(\bar{\eta}_{t,s}(x))|\Gamma_{t,s}(x)]^2 \, dx$$

$$= \frac{1}{[2\pi(t-s)]^{d/2}} \int\int |q(x - \rho(x)[w - h(x)(t-s)] + \rho_0(x)(t-s))|^2$$

$$\times \exp\{2h^*(x)w - |h(x)|^2(t-s) - 2\alpha^*(x)[w - h(x)(t-s)]$$

$$+2\bar{\alpha}(x)(t-s)+2\alpha_0(x)(t-s)\}\exp\left\{-\frac{|w|^2}{2(t-s)}\right\}dw\,dx.$$

Next

$$2[h(x)-\alpha(x)]^*w-\frac{|w|^2}{2(t-s)}=2|h(x)-\alpha(x)|^2(t-s)-\frac{|w-2[h(x)-\alpha(x)](t-s)|^2}{2(t-s)},$$

so that, using the new variables (x,v) with $v=w-2[h(x)-\alpha(x)](t-s)$

$$\mathbf{E}\dagger|\bar{Q}_t^s q|^2\leqq e^{C(t-s)}\frac{1}{[2\pi(t-s)]^{d/2}}\int\int|q(x-\rho(x)v$$

$$+\gamma(x)(t-s))|^2\exp\left\{-\frac{|v|^2}{2(t-s)}\right\}dv\,dx,$$

where $\gamma(x)\triangleq\rho_0(x)-\rho(x)[h(x)-2\alpha(x)]$.

In the particular case where $\rho(x)\equiv\rho$, the application $F(x)\triangleq x-\rho v+\gamma(x)(t-s)$ is a diffeomorphism provided $0\leqq(t-s)<1/C$, and moreover the Jacobian is bounded below by $[1-C(t-s)]$. Therefore, using the new variables (y,z) with $y=F(x)$

$$\mathbf{E}\dagger|\bar{Q}_t^s q|^2\leqq\frac{e^{C(t-s)}}{1-C(t-s)}\frac{1}{[2\pi(t-s)]^{d/2}}\int\int|q(y)|^2\exp\left\{-\frac{|v|^2}{2(t-s)}\right\}dv\,dy$$

$$\leqq\frac{e^{C(t-s)}}{1-C(t-s)}\int|q(y)|^2\,dy,$$

provided $0\leqq(t-s)\leqq\delta\leqq1/C$, which finishes the proof. □

Remark A.1 According to the detail of the proof above, it is enough for the Condition (A) to hold, that

$$\frac{1}{[2\pi(t-s)]^{d/2}}\int\int|q(x-\rho(x)v+\gamma(x)(t-s))|^2\exp\left\{-\frac{|v|^2}{2(t-s)}\right\}dv\,dx\leqq e^{C(t-s)}\int|q(y)|^2\,dy,$$

for any bounded function γ.

Proof of Proposition 5.6 **Here again, it is enough to prove the result for $n=0$ and $|\alpha|=1$. Throughout the proof, the summation convention over repeated indices j is used.**

For q smooth enough, it holds

$$\frac{\partial}{\partial x_i}\bar{Q}_t^s q(x)=\frac{\partial q}{\partial x_j}(\bar{\eta}_{t,s}(x))\Bigg[\delta^{i,j}-\sum_{k=1}^{d}\frac{\partial \rho_k^j}{\partial x_i}(x)[Y_t^k-Y_s^k-(t-s)h_k(x)]$$

$$+(t-s)\left(\sum_{k=1}^{d}\rho_k^j(x)\frac{\partial h_k}{\partial x_i}(x)+\frac{\partial \rho_0^j}{\partial x_i}(x)\right)\Bigg]\Gamma_{t,s}(x)$$

$$+q(\bar{\eta}_{t,s}(x))\Bigg[\sum_{k=1}^{d}\left(\frac{\partial h_k}{\partial x_i}(x)-\frac{\partial \alpha_k}{\partial x_i}(x)\right)[Y_t^k-Y_s^k-(t-s)h_k(x)]$$

$$+(t-s)\left(\sum_{k=1}^{d}\alpha_k(x)\frac{\partial h_k}{\partial x_i}(x)+\frac{\partial \bar{\alpha}}{\partial x_i}(x)+\frac{\partial \alpha_0}{\partial x_i}(x)\right)\Bigg]\Gamma_{t,s}(x)$$

$$=\bar{Q}_t^s\frac{\partial q}{\partial x_i}(x)-\sum_{k=1}^{d}\frac{\partial \rho_k^j}{\partial x_i}(x)[Y_t^k-Y_s^k-(t-s)h_k(x)]\bar{Q}_t^s\frac{\partial q}{\partial x_j}(x)$$

$$+(t-s)\left(\sum_{k=1}^{d}\rho_k^j(x)\frac{\partial h_k}{\partial x_i}(x)+\frac{\partial \rho_0^j}{\partial x_i}(x)\right)\bar{Q}_t^s\frac{\partial q}{\partial x_j}(x)$$

$$+\sum_{k=1}^{d}\left(\frac{\partial h_k}{\partial x_i}(x)-\frac{\partial \alpha_k}{\partial x_i}(x)\right)[Y_t^k-Y_s^k-(t-s)h_k(x)]\bar{Q}_t^s q(x)$$

$$+(t-s)\left(\sum_{k=1}^{d}\alpha_k(x)\frac{\partial h_k}{\partial x_i}(x)+\frac{\partial \bar{\alpha}}{\partial x_i}(x)+\frac{\partial \alpha_0}{\partial x_i}(x)\right)\bar{Q}_t^s q(x).$$

Therefore, under Condition (A)

$$\mathbf{E}^\dagger\left|\frac{\partial}{\partial x_i}\bar{Q}_t^s q-\bar{Q}_t^s\frac{\partial q}{\partial x_i}\right|^2\leqq C(t-s)\left[\mathbf{E}^\dagger\sum_{j=1}^{m}\left|\bar{Q}_t^s\frac{\partial q}{\partial x_j}\right|^2+\mathbf{E}^\dagger|\bar{Q}_t^s q|^2\right]$$

$$\leqq C(t-s)\left[\sum_{j=1}^{m}\left|\frac{\partial q}{\partial x_j}\right|^2+|q|^2\right]$$

$$\leqq C(t-s)\|q\|^2. \qquad \square$$

PARTICLE APPROXIMATION FOR FIRST ORDER STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract

A class of degenerate second order stochastic PDE is considered, for which a representation result in terms of stochastic characteristics has been proved by Krylov–Rozovskii [2] and Kunita [3,4]. An example of a stochastic PDE in this class has been exhibited in Florchinger–LeGland [1] as the result of a Trotter–like product formula for the Zakai equation of diffusion processes observed in correlated noise. Particle approximations are introduced for this class of stochastic PDE, and error estimates are provided which extend the results of Raviart [6] on first order deterministic PDE.

1 Introduction

Consider the following stochastic differential equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,[dW_t - e(X_t)\,dt]\ , \tag{1.1}$$

where $\{W_t\,,\,t \geq 0\}$ is a d–dimensional standard Wiener process, and the associated stochastic flow of diffeomorphisms $\{\xi_{s,t}(\cdot)\,,\,0 \leq s \leq t\}$, and define

$$\Xi_{0,t}(x) \triangleq \exp\left\{\int_0^t e^*(\xi_{0,s}(x))\,dW_s - \tfrac{1}{2}\int_0^t |e(\xi_{0,s}(x))|^2\,ds + \int_0^t c(\xi_{0,s}(x))\,ds\right\}\ .$$

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Introduce the following partial differential operators

$$L \triangleq \tfrac{1}{2} \sum_{i,j=1}^{m} a^{i,j} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b^i \frac{\partial}{\partial x_i} + c \ ,$$

$$B_k \triangleq e_k + \sum_{i=1}^{m} \sigma_k^i \frac{\partial}{\partial x_i} \ , \qquad 1 \leq k \leq d \ ,$$

with $a = \sigma\, \sigma^*$, and the stochastic PDE

$$dq_t = L^* q_t \, dt + \sum_{k=1}^{d} B_k^* q_t \, dW_t^k \ . \tag{1.2}$$

Because of the relation $a = \sigma\, \sigma^*$ between coefficients of higher order partial derivatives in operators L and B_k, equation (1.2) is a degenerate second order stochastic PDE or equivalently, after transformation into Stratonovich form, a first order stochastic PDE. Existence and representation results have been obtained by Kunita [4] for (generally nonlinear) first order stochastic PDE, based on the notion of stochastic characteristics.

In a previous work [1], the Zakai equation for the nonlinear filtering of diffusion processes observed in correlated noise has been considered. A decomposition of the Zakai equation has been introduced, exhibiting a degenerate second order stochastic PDE similar to (1.2) in the *correction step*. In addition, a time discretization scheme has been proposed for this degenerate second order stochastic PDE, with rate of convergence of order $\sqrt{\delta}$, where δ is the time step.

The purpose of this paper is to provide a discretization scheme of the degenerate second order stochastic PDE (1.2) with respect to the space variable $x \in \mathbf{R}^m$. This approximation relies on the representation of the solution in terms of stochastic characteristics, and approximation of the initial condition by a convex linear combination of Dirac masses. This kind of aproximation is called a *particle approximation*, see Raviart [6].

More specifically, for any probability measure $\mu(dx)$ on $\mathbf{R}^m$, define the transformed measure $Q_t\, \mu(dx)$ by

$$\langle Q_t\, \mu, \phi \rangle = \int \phi(\xi_{0,t}(x))\, \Xi_{0,t}(x)\, \mu(dx) \ , \tag{1.3}$$

for any test function ϕ, or equivalently

$$Q_t\, \mu(A) = \int_{\xi_{0,t}^{-1}(A)} \Xi_{0,t}(x)\, \mu(dx) \ .$$

Note that, if ϕ is regular enough, then the Itô formula gives

$$d[\, \phi(\xi_{0,t}(x)\, \Xi_{0,t}(x) \,] = L\phi(\xi_{0,t}(x)) \cdot \Xi_{0,t}(x)\, dt + \sum_{k=1}^{d} B_k \phi(\xi_{0,t}(x)) \cdot \Xi_{0,t}(x)\, dW_t^k \ .$$

Therefore $\mu_t(dx) = Q_t\,\mu(dx)$ solves equation (1.2) in weak form, i.e.

$$d\mu_t = L^*\mu_t\,dt + \sum_{k=1}^{d} B_k^*\mu_t\,dW_t^k\ , \qquad \mu_0 = \mu\ . \tag{1.4}$$

Consider next the following two different assumptions on the original measure $\mu_0(dx)$:

□ Assume that the original measure $\mu(dx)$ has a density $q(x)$ with respect to the Lebesgue measure on $\mathbf{R}^m$, i.e. $\mu(dx) = q(x)\,dx$. Then, the transformed measure $Q_t\,\mu(dx)$ has itself a density $q_t(x)$ which satisfies

$$q_t(\xi_{0,t}(x)) \cdot J_{0,t}(x) = \Xi_{0,t}(x) \cdot q(x)\ ,$$

or in integrated form

$$\int_A q_t(x)\,dx = \int_{\xi_{0,t}^{-1}(A)} \Xi_{0,t}(x) \cdot q(x)\,dx\ .$$

Here, $J_{0,t}(\cdot)$ is the Jacobian (i.e. the determinant of the Jacobian matrix) of the stochastic flow $\xi_{0,t}(\cdot)$. In addition, the density $q_t(x)$ solves the degenerate second order stochastic PDE

$$dq_t = L^*q_t\,dt + \sum_{k=1}^{d} B_k^*q_t\,dW_t^k\ , \qquad q_0 = q\ . \tag{1.5}$$

□ Assume that the original measure $\mu(dx)$ is a convex linear combination of Dirac masses, also called *particles*

$$\mu(dx) = \sum_{i\in I} a^i\,\delta(x - x^i)\ ,$$

where $\{a^i\,,\, i \in I\}$ are the particle weights, and $\{x^i\,,\, i \in I\}$ are the particle locations. Then, the transformed measure $Q_t\,\mu(dx)$ has a similar representation

$$Q_t\mu(dx) = \sum_{i\in I} a_t^i\,\delta(x - x_t^i)\ ,$$

where the particles have been transported by the flow i.e. $x_t^i = \xi_{0,t}(x^i)$, and the weights have been updated according to $a_t^i = a^i\,\Xi_{0,t}(x^i)$.

The idea behind particle approximation for equation (1.2) is the following :

- given an initial condition $\mu_0(dx)$ with density $q_0(x)$, find an approximation $\mu_0^h(dx)$ in terms of a linear convex combination of Dirac masses,
- use the exact solution of equation (1.4) with the approximation $\mu_0^h(dx)$ as initial condition, as an approximation for the solution of the original equation (1.5), and get error estimate if possible.

This can be illustrated by the following diagram

$$
\begin{tikzcd}
q_0(x)\,dx = \mu_0(dx) \arrow[r] \arrow[d, "Q_t"'] & \mu_0^h(dx) \arrow[d, "Q_t"] \\
q_t(x)\,dx = \mu_t(dx) \arrow[r] & \mu_t^h(dx)
\end{tikzcd}
$$

The remaining of this section is devoted to recalling standard results concerning stochastic flows of diffeomorphisms and stochastic PDE.

Proposition 1.1 *Let $n \geq 0$ be fixed. Assume that*

- *b, σ and e have bounded derivatives up to order $(n+1)$.*
- *c has bounded derivatives up to order n.*

Then $\xi_{s,t}(\cdot)$ is a C^n-diffeomorphism in $\mathbf{R}^m$. In addition, the following estimates hold for all $p \geq 1$

$$\sup_{x \in \mathbf{R}^m} \mathbf{E}\left[|D^\alpha \xi_{s,t}(x)|^p \right] < \infty \;, \qquad 1 \leq |\alpha| \leq n \;,$$

$$\sup_{x \in \mathbf{R}^m} \mathbf{E}\left[|D^\alpha \Xi_{s,t}(x)|^p \right] < \infty \;, \qquad 0 \leq |\alpha| \leq n \;.$$

Restricting to compact sets of $\mathbf{R}^m$, it is possible to invert the supremum and the mathematical expectation in the estimates above, see the Corollary 4.6.7 of Kunita [5]

Proposition 1.2 *Under the assumptions of the Proposition 1.1, there exists a constant $C > 0$, such that for any compact set $B \subset \mathbf{R}^m$ and $\varepsilon > 0$ the following uniform estimates hold for all $p \geq 1$*

$$\mathbf{E}\left[\sup_{x \in B} |D^\alpha \xi_{s,t}(x)|^p \right] \leq C\ [1 + \delta^{p-\varepsilon}] \;, \qquad 1 \leq |\alpha| \leq n \;,$$

$$\mathbf{E}\left[\sup_{x \in B} |D^\alpha \Xi_{s,t}(x)|^p \right] \leq C\ [1 + \delta^{p-\varepsilon}] \;, \qquad 0 \leq |\alpha| \leq n \;,$$

where $\delta = \delta(B)$ denotes the diameter of B.

For all $n \geq 0$, $p \geq 1$, let $W^{n,p} \equiv W^{n,p}(\mathbf{R}^m)$ denote the space of real-valued Lebesgue-measurable functions on $\mathbf{R}^m$ whose generalized derivatives up to order n are integrable in p-mean, and define the corresponding norm $\|\cdot\|_{n,p}$ and semi-norm $|\cdot|_{n,p}$ by

$$\|u\|_{n,p}^p \triangleq \sum_{0\leq|\alpha|\leq n} \int |D^\alpha u(x)|^p \, dx \qquad \text{and} \qquad |u|_{n,p}^p \triangleq \sum_{|\alpha|=n} \int |D^\alpha u(x)|^p \, dx \; ,$$

respectively.

Consider the following degenerate second order stochastic PDE

$$dq_t = L^* q_t \, dt + \sum_{k=1}^{d} B_k^* q_t \, dW_t^k \; , \qquad q_0 = q \; . \tag{1.6}$$

Although no coercivity hypothesis is satisfied, the following existence, uniqueness and regularity result is proved in Krylov–Rozovskii [2].

Theorem 1.3 *Let $n \geq 1$ be fixed. Assume that*

- *a has bounded derivatives up to order $\max(n,2)$,*
- *b, σ, c and e have bounded derivatives up to order n,*
- *the initial condition satisfies $q_0 \in W^{n,p}$.*

Then equation (1.6) has a unique solution $q \in M^p(0,T\,;W^{n,p})$. In addition

$$q \in L^p(\Omega\,; C_w([0,T]\,; W^{n,p})),$$

and the following estimate holds

$$\mathbf{E}[\sup_{0\leq t\leq T} \|q_t\|_{n,p}^p] \leq \|q_0\|_{n,p}^p \, e^{CT} \; .$$

2 Quadrature–based particle approximation

With the quadrature formula (A.1)

$$\int g(x)\, dx \sim \sum_{i\in I} \omega^i \, g(x^i) \; ,$$

is associated the following particle approximation for the initial density $q_0(x)$

$$q_0(x)\, dx = \mu_0(dx) \sim \mu_0^h(dx) = \sum_{i\in I} \omega^i \, q_0(x^i) \, \delta(x - x^i) \; . \tag{2.1}$$

This induces the following particle approximation for the solution $q_t(x)$ of equation (1.6)

$$q_t(x)\, dx = \mu_t(dx) \sim \mu_t^h(dx) = \sum_{i\in I} \omega^i \, \Xi_{0,t}(x^i) \, q_0(x^i) \, \delta(x - \xi_{0,t}(x^i)) \; .$$

The following error estimate holds in Sobolev space with negative exponent, which extends the result of Raviart to the case of first order stochastic PDE.

Theorem 2.1 *Let $n \geq m$ be fixed. Assume that*

- *b, σ, c and e have bounded derivatives up to order $(n+1)$,*
- *the initial condition satisfies $q_0 \in W^{n,p}$.*

Then there exists a constant $C > 0$ independent of h, such that

$$\mathbf{E}\|\mu_t - \mu_t^h\|_{-n,p} \leq C\, h^n\, \|q_0\|_{n,p} \; .$$

PROOF. Let $\phi \in W^{n,p'}$ be an arbitrary test function. Since

$$\langle \mu_t, \phi \rangle = \int \phi(\xi_{0,t}(x))\, \Xi_{0,t}(x)\, q_0(x)\, dx \;, \qquad \langle \mu_t^h, \phi \rangle = \sum_{i \in I} \omega^i\, \phi(\xi_{0,t}(x^i))\, \Xi_{0,t}(x^i)\, q_0(x^i) \;,$$

it follows from Theorem A.2 that

$$|\langle \mu_t, \phi \rangle - \langle \mu_t^h, \phi \rangle| \leq C\, h^n\, |g|_{n,1} \;,$$

with $g = \phi \circ \xi_{0,t} \cdot \Xi_{0,t}\, q_0$, provided $g \in W^{n,1}$, $n \geq m$.

Under the assumptions on the coefficients, $\phi \circ \xi_{0,t} \in W^{n,p'}$ and $\Xi_{0,t} \cdot q_0 \in W^{n,p}$, for conjugate p and p'. Moreover, the generalized Leibniz formula yields

$$|g|_{n,1} \leq \sum_{(\alpha,\beta) \in I_n} \int |\chi_{\alpha,\beta}(x)\, D^\alpha \phi(\xi_{0,t}(x))\, D^\beta q_0(x)|\, dx \;,$$

where I_n denotes the set of pairs (α, β) of multi-indices such that $|\alpha| + |\beta| \leq n$, and $\chi_{\alpha,\beta}(\cdot)$ are random fields involving the derivatives of $\xi_{0,t}(\cdot)$ and $\Xi_{0,t}(\cdot)$ up to order n. Using back and forth the changes of variable induced by the differomorphisms $\xi_{0,t}(\cdot)$ and $\xi_{0,t}^{-1}(\cdot)$, and the Hölder inequality, gives

$$\begin{aligned}
|g|_{n,1} &\leq \sum_{(\alpha,\beta) \in I_n} \int |\chi_{\alpha,\beta}(\xi_{0,t}^{-1}(x))\, D^\alpha \phi(x)\, D^\beta q_0(\xi_{0,t}^{-1}(x))|\, [J_{0,t}(\xi_{0,t}^{-1}(x))]^{-1}\, dx \\
&\leq \sum_{(\alpha,\beta) \in I_n} \left\{ \int |D^\alpha \phi(x)|^{p'}\, dx \right\}^{1/p'} \left\{ \int |\chi_{\alpha,\beta}(\xi_{0,t}^{-1}(x))\, D^\beta q_0(\xi_{0,t}^{-1}(x))|^p \right. \\
&\qquad\qquad \left. [J_{0,t}(\xi_{0,t}^{-1}(x))]^{-p}\, dx \right\}^{1/p} \\
&\leq \|\phi\|_{n,p'} \sum_{(\alpha,\beta) \in I_n} \left\{ \int |\chi_{\alpha,\beta}(x)\, D^\beta q_0(x)|^p\, [J_{0,t}(x)]^{-(p-1)}\, dx \right\}^{1/p} .
\end{aligned}$$

Therefore

$$\frac{|\langle \mu_t, \phi \rangle - \langle \mu_t^h, \phi \rangle|}{\|\phi\|_{n,p'}} \leq C\, h^n \sum_{(\alpha,\beta) \in I_n} \left\{ \int |\chi_{\alpha,\beta}(x)\, D^\beta q_0(x)|^p\, [J_{0,t}(x)]^{-(p-1)}\, dx \right\}^{1/p} ,$$

and

$$\mathbf{E}\|\mu_t - \mu_t^h\|_{-n,p} \leq C\, h^n \sum_{(\alpha,\beta)\in I_n} \left\{ \int \mathbf{E}\left\{ |\chi_{\alpha,\beta}(x)|^p \, [J_{0,t}(x)]^{-(p-1)} \right\} \right.$$

$$\left. |D^\beta q_0(x)|^p \, dx \right\}^{1/p} .$$

From estimates in Proposition 1.1, it holds

$$\sup_{x\in\mathbf{R}^m} \mathbf{E}\left\{ |\chi_{\alpha,\beta}(x)|^p \, [J_{0,t}(x)]^{-(p-1)} \right\} < \infty ,$$

so that

$$\mathbf{E}\|\mu_t - \mu_t^h\|_{-n,p} \leq C\, h^n \, \|q_0\|_{n,p} . \qquad \square$$

Regularization

Let $\zeta(x)$ be a continuous cut–off function defined on $\mathbf{R}^m$, which satisfies

$$(i) \qquad \int \zeta(x)\, dx = 1 ,$$

$$(ii) \qquad \int x^\alpha \, \zeta(x)\, dx = 0 , \qquad 1 \leq |\alpha| \leq k-1 ,$$

$$(iii) \qquad \int |x|^k \, |\zeta(x)|\, dx < \infty ,$$

for some $k \geq 2$. For any $\varepsilon > 0$, $\zeta_\varepsilon(x)$ is defined by the following scaling

$$\zeta_\varepsilon(x) \triangleq \frac{1}{\varepsilon^m}\, \zeta(\frac{x}{\varepsilon}) .$$

With the particle approximation

$$\mu_t^h(dx) = \sum_{i\in I} \omega^i \, \Xi_{0,t}(x^i)\, q_0(x^i)\, \delta(x - x_t^i) ,$$

is associated the regularized measure

$$\mu_t^{h,\varepsilon}(dx) = \mu_t^h * \zeta_\varepsilon(dx) = q_t^{h,\varepsilon}(x)\, dx ,$$

where the density $q_t^{h,\varepsilon}(x)$ is given by

$$q_t^{h,\varepsilon}(x) = \sum_{i\in I} \omega^i \, \Xi_{0,t}(x^i)\, q_0(x^i)\, \zeta_\varepsilon(x - x_t^i) .$$

The main result of this section is the following theorem, which is an extension of the Theorem 4.2 in [6], to the case of first order stochastic PDE.

Theorem 2.2 *Let $n > m$ be fixed. Assume that*

- *the cut-off function ζ satisfies (i)–(iii) for some $k \geq 2$, and $\zeta \in W^{n,1}$,*
- *b, σ, c and e have bounded derivatives up to order $(\ell + 1)$,*
- *the initial condition satisfies $q_0 \in W^{\ell,p}$,*

where $\ell = \max(k, n)$.

Then, there exists a constant C independent of both h and ε, such that

$$\left\{\mathbf{E}\|q_t - q_t^{h,\varepsilon}\|_{0,p}^p\right\}^{1/p} \leq C \left\{\varepsilon^k \|q_0\|_{k,p} + (h/\varepsilon)^n \|q_0\|_{n,p}\right\} .$$

PROOF. Obviously

$$q_t - q_t^{h,\varepsilon} = [q_t - q_t * \zeta_\varepsilon] + [q_t * \zeta_\varepsilon - q_t^{h,\varepsilon}] .$$

First, it follows from Lemma 4.4 in [6] that

$$\|q_t - q_t * \zeta_\varepsilon\|_{0,p} \leq C\ \varepsilon^k\ |q_t|_{k,p}$$

provided $q_t \in W^{k,p}$. Under the assumptions, Theorem 1.3 gives

$$\left\{\mathbf{E}\|q_t - q_t * \zeta_\varepsilon\|_{0,p}^p\right\}^{1/p} \leq C\ \varepsilon^k\ \left\{\mathbf{E}|q_t|_{k,p}^p\right\}^{1/p} \leq C\ \varepsilon^k\ \|q_0\|_{k,p} .$$

On the other hand, using the change of variable induced by the diffeomorphism $\xi_{0,t}^{-1}(\cdot)$, it holds for all $x \in \mathbf{R}^m$

$$\begin{aligned} q_t * \zeta_\varepsilon(x) - q_t^{h,\varepsilon}(x) &= \int \Xi_{0,t}(z)\ q_0(z)\ \zeta_\varepsilon(x - \xi_{0,t}(z))\, dz \\ &\quad - \sum_{i \in I} \omega^i\, \Xi_{0,t}(x^i)\, q_0(x^i)\ \zeta_\varepsilon(x - \xi_{0,t}(x^i)) = E(g(x,\cdot)) \end{aligned}$$

with $g(x,\cdot) = \Xi_{0,t}\ q_0 \cdot \zeta_\varepsilon(x - \xi_{0,t})$. Therefore, it follows from Theorem A.1 that for all $x \in \mathbf{R}^m$

$$|q_t * \zeta_\varepsilon(x) - q_t^{h,\varepsilon}(x)| \leq C\ h^n\ |g(x,\cdot)|_{n,1}$$

provided $g(x,\cdot) \in W^{n,1}$, $n \geq m$. Moreover, the generalized Leibniz formula yields

$$|g(x,\cdot)|_{n,1} \leq \sum_{(\alpha,\beta) \in I_n} \int |\chi'_{\alpha,\beta}(z)\ D^\beta q_0(z) D^\alpha \zeta_\varepsilon(x - \xi_{0,t}(z))|\, dx ,$$

where I_n denotes the set of pairs (α, β) of multi-indices such that $|\alpha| + |\beta| \leq n$, and $\chi'_{\alpha,\beta}(\cdot)$ are random fields involving the derivatives of $\xi_{0,t}(\cdot)$ and $\Xi_{0,t}(\cdot)$ up to order n. From the technical lemma below, it follows that

$$\begin{aligned} \int |g(x,\cdot)|_{n,1}^p\, dx \leq C \sum_{(\alpha,\beta) \in I_n} \left\{\int |D^\alpha \zeta_\varepsilon(x)|\, dx\right\}^p \Big\{\int |\chi'_{\alpha\beta}(x)\ D^\beta q_0(x)|^p \\ [J_{0,t}(x)]^{-(p-1)}\, dx\Big\} . \end{aligned}$$

Making use of

$$D^\alpha \zeta_\varepsilon(x) = \frac{1}{\varepsilon^{m+|\alpha|}} D^\alpha \zeta(\frac{x}{\varepsilon}) ,$$

taking mathematical expectation on both sides, and raising to the power $1/p$ gives

$$\left\{\mathbf{E} \int |g(x,\cdot)|_{n,1}^p \, dx\right\}^{1/p} \leq C \frac{1}{\varepsilon^n} \|\zeta\|_{n,1} \sum_{(\alpha,\beta)\in I_n} \left\{\int \mathbf{E}\left\{|\chi'_{\alpha,\beta}(x)|^p \, [J_{0,t}(x)]^{-(p-1)}\right\} |D^\beta q_0(x)|^p \, dx\right\}^{1/p} .$$

From estimates in Proposition 1.1, it holds

$$\sup_{x\in\mathbf{R}^m} \mathbf{E}\left\{|\chi'_{\alpha,\beta}(x)|^p \, [J_{0,t}(x)]^{-(p-1)}\right\} < \infty .$$

Therefore

$$\begin{aligned}\left\{\mathbf{E}\|q_t * \zeta_\varepsilon - q_t^{h,\varepsilon}\|_{0,p}^p\right\}^{1/p} &\leq C \, h^n \left\{\mathbf{E} \int |g(x,\cdot)|_{n,1}^p \, dx\right\}^{1/p} \\ &\leq C \, (h/\varepsilon)^n \, \|\zeta\|_{n,1} \, \|q_0\|_{n,p} . \qquad \square\end{aligned}$$

Lemma 2.3 *Let $f \in L^p$ and $g \in L^1$, and define*

$$I(x) = \int f(z) \, g(x - \xi_{0,t}(z)) \, dz .$$

Then $I \in L^p$ and in addition

$$\left\{\int |I(x)|^p \, dx\right\}^{1/p} \leq \left\{\int |f(x)|^p \, [J_{0,t}(x)]^{-(p-1)} \, dx\right\}^{1/p} \int |g(x)| \, dx .$$

PROOF. Using back and forth the changes of variable induced by the differomorphisms $\xi_{0,t}(\cdot)$ and $\xi_{0,t}^{-1}(\cdot)$, and the Lemma 4.3 in [6], gives

$$I(x) = \int f(\xi_{0,t}^{-1}(z)) \, [J_{0,t}(\xi_{0,t}^{-1}(z))]^{-1} \, g(x - z) \, dz ,$$

and

$$\begin{aligned}\left\{\int |I(x)|^p \, dx\right\}^{1/p} &\leq \left\{\int |f(\xi_{0,t}^{-1}(x))|^p \, [J_{0,t}(\xi_{0,t}^{-1}(x)]^{-p} \, dx\right\}^{1/p} \int |g(x)| \, dx \\ &\leq \left\{\int |f(x)|^p \, [J_{0,t}(x)]^{-(p-1)} \, dx\right\}^{1/p} \int |g(x)| \, dx . \qquad \square\end{aligned}$$

3 Adapted particle approximation

Consider the particle approximation (A.3) for the initial condition $\mu_0(dx)$

$$\mu_0(dx) \sim \mu_0^h(dx) = \sum_{i \in I} a^i \, \delta(x - x^i) \; ,$$

where the particle weights $\{a^i \, , \, i \in I\}$ and the particle locations $\{x^i \, , \, i \in I\}$ are defined in the following way

$$a^i \triangleq \mu_0(B^i) = \int_{B^i} \mu_0(dx) \; , \qquad x^i \triangleq \frac{1}{a^i} \int_{B^i} x \, \mu_0(dx) \; ,$$

depending on the measure $\mu_0(dx)$. This induces the following particle approximation for the solution $\mu_t(x)$ of equation (1.4)

$$\mu_t(dx) \sim \mu_t^h(dx) = \sum_{i \in I} a^i \, \Xi_{0,t}(x^i) \, \delta(x - \xi_{0,t}(x^i)) \; .$$

Parallel to the Theorem 2.1 above, the following error estimate holds in Sobolev space with negative exponent.

Theorem 3.1 *Assume that*

- *b, σ, c and e have bounded derivatives up to order 3,*
- *for all $i \in I$, the set $B^i \subset \mathbf{R}^m$ is compact.*

Then there exists a constant $C > 0$, such that

$$\mathbf{E} \| \mu_t - \mu_t^h \|_{-2,1} \leq C \sum_{i \in I} \delta_i^2 \, a^i \; ,$$

where $a^i = \mu_0(B^i)$ and $\delta_i = \delta(B^i)$ denotes the diameter *of the set B^i.*

PROOF. Let $\phi \in W^{2,\infty}$ be an arbitrary test function. Since

$$\langle \mu_t, \phi \rangle = \int \phi(\xi_{0,t}(x)) \, \Xi_{0,t}(x) \, \mu_0(dx) \; , \qquad \langle \mu_t^h, \phi \rangle = \sum_{i \in I} a^i \, \phi(\xi_{0,t}(x^i)) \, \Xi_{0,t}(x^i) \; ,$$

it follows from estimate (A.6) that

$$| \langle \mu_t, \phi \rangle - \langle \mu_t^h, \phi \rangle | \leq \tfrac{1}{2} \sum_{i \in I} |g|_{2,\infty,\bar{B}^i} \, \delta_i^2 \, a^i \; ,$$

with $g = \phi \circ \xi_{0,t} \cdot \Xi_{0,t}$, where $\widehat{B}^i$ denotes the *convex hull* of B^i. The generalized Leibniz formula yields

$$\begin{aligned}
|g|_{2,\infty,B} &\leq \sum_{|\alpha|\leq 2} \sup_{x\in B} |\chi_\alpha(x)\, D^\alpha \phi(\xi_{0,t}(x))| \\
&\leq \sum_{|\alpha|\leq 2} \left[\sup_{x\in B} |\chi_\alpha(x)|\right] \left[\sup_{x\in \mathbf{R}^m} |D^\alpha \phi(x)|\right] \\
&\leq \|\phi\|_{2,\infty} \sum_{|\alpha|\leq 2} \sup_{x\in B} |\chi_\alpha(x)| \; ,
\end{aligned}$$

where $\chi_\alpha(\cdot)$ are random fields involving the derivatives of $\xi_{0,t}(\cdot)$ and $\Xi_{0,t}(\cdot)$ up to order 2. Therefore

$$\frac{|\langle \mu_t, \phi\rangle - \langle \mu_t^h, \phi\rangle|}{\|\phi\|_{2,\infty}} \leq \tfrac{1}{2} \sum_{i\in I} \sum_{|\alpha|\leq 2} \sup_{x\in \widehat{B}^i} |\chi_\alpha(x)|\, \delta_i^2\, a^i \; ,$$

and

$$\mathbf{E}\|\mu_t - \mu_t^h\|_{-2,1} \leq \tfrac{1}{2} \sum_{i\in I} \sum_{|\alpha|\leq 2} \mathbf{E}\left[\sup_{x\in \widehat{B}^i} |\chi_\alpha(x)|\right] \delta_i^2\, a^i \; .$$

From estimates in Proposition 1.2, it holds

$$\mathbf{E}\left[\sup_{x\in \widehat{B}^i} |\chi_\alpha(x)|\right] \leq C\, \left[1 + \delta_i^{2-\epsilon}\right] \; ,$$

for some p, where $\delta_i = \delta(B^i)$ denotes the diameter of both B^i and its convex hull $\widehat{B}^i$, so that

$$\mathbf{E}\|\mu_t - \mu_t^h\|_{-2,1} \leq C \sum_{i\in I} \left[1 + \delta_i^{2-\epsilon}\right] \delta_i^2\, a^i \; . \qquad \Box$$

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A Particle approximation of functions

Consider the following quadrature formula on $\mathbf{R}^m$

$$\int g(x)\,dx \sim \sum_{i\in I} \omega^i\, g(x^i)\ , \tag{A.1}$$

where $\{x^i\,,\, i \in I\}$ is a coordinate grid of size $h > 0$. $I = \mathbf{Z}^m$ and $\omega^i = h^m$ is the Lebesgue measure of the m-dimensional cube B^i with center x^i and edge size h. For all $g \in C(\mathbf{R}^m)$, the quadrature error associated with the quadrature formula (A.1) is defined by

$$E_i(g) \triangleq \int_{B^i} g(x)\,dx - \omega^i\, g(x^i)\ , \qquad E(g) \triangleq \sum_{i\in I} E_i(g)\ .$$

The following estimate is proved in Raviart [6]

Theorem A.1 *There is a constant $C > 0$ independent of h such that*

$$|E(g)| \leq C\ h^n\ |g|_{n,1}\ ,$$

for all $g \in W^{n,1}$, $n \geq m$.

Let $\mu(dx)$ be a probability measure on $\mathbf{R}^m$ having a continuous density $q(x)$ with respect to the Lebesgue measure, i.e. $\mu(dx) = q(x)\,dx$. With the quadrature formula (A.1) is associated the following particle approximation for the density $q(x)$

$$q(x)\,dx = \mu(dx) \sim \mu^h(dx) = \sum_{i\in I} \omega^i\, q(x^i)\, \delta(x - x^i)\ , \tag{A.2}$$

so that, for any test function ϕ

$$\langle \mu, \phi \rangle = \int \phi(x)\, q(x)\,dx\ , \qquad \langle \mu^h, \phi \rangle = \sum_{i\in I} \omega^i\, \phi(x^i)\, q(x^i)\ .$$

The following result is proved in Raviart [6]

Theorem A.2 *There is a constant $C > 0$ independent of h such that*

$$\|\mu - \mu^h\|_{-n,p} \leq C\ h^n\ \|q\|_{n,p}\ ,$$

for all $q \in W^{n,p}$, $n \geq m$.

PROOF. From Theorem A.1, it holds

$$|\langle\mu,\phi\rangle - \langle\mu^h,\phi\rangle| = |E(g)| \leq C\ h^n\ |g|_{n,1}\ ,$$

with $g = \phi \cdot q$, provided $g \in W^{n,1}$, $n \geq m$. The generalized Leibniz formula and the Hölder inequality yield

$$|g|_{n,1} \leq C\ \|\phi\|_{n,p'}\ \|q\|_{n,p}\ ,$$

for conjugate p and p', and therefore

$$\|\mu - \mu^h\|_{-n,p} = \sup_{\phi \in W^{n,p'}} \frac{|\langle\mu,\phi\rangle - \langle\mu^h,\phi\rangle|}{\|\phi\|_{n,p'}} \leq C\ h^n\ \|q\|_{n,p}\ . \qquad \square$$

Another possible approximation is to consider a partition $\{B^i\,,\, i \in I\}$ of $\mathbf{R}^m$, and to define the following particle approximation for the probability measure $\mu(dx)$

$$\mu(dx) \sim \mu^h(dx) = \sum_{i \in I} a^i\ \delta(x - x^i)\ , \tag{A.3}$$

where the particle weights $\{a^i\,,\, i \in I\}$ and the particle locations $\{x^i\,,\, i \in I\}$ are defined in the following way

$$a^i \triangleq \mu(B^i) = \int_{B^i} \mu(dx)\ , \qquad x^i \triangleq \frac{1}{a^i} \int_{B^i} x\ \mu(dx)\ , \tag{A.4}$$

depending on the measure $\mu(dx)$ so that, for any test function ϕ

$$\langle\mu,\phi\rangle = \int \phi(x)\,\mu(dx)\ , \qquad \langle\mu^h,\phi\rangle = \sum_{i \in I} a^i\ \phi(x^i)\ .$$

For all $\phi \in C(\mathbf{R}^m)$, the quadrature error associated with the formula (A.3), is defined by

$$E_i'(\phi) \triangleq \int_{B^i} \phi(x)\,\mu(dx) - a^i\ \phi(x^i)\ , \qquad E'(\phi) \triangleq \sum_{i \in I} E_i'(\phi)\ .$$

Parallel to the Theorem A.2 above, the following result holds

Theorem A.3 *For any partition* $\{B^i\,,\, i \in I\}$

$$\|\mu - \mu^h\|_{-2,1} \leq \tfrac{1}{2} \sum_{i \in I} \delta_i^2\ a^i\ , \tag{A.5}$$

where $a^i = \mu(B^i)$ *and* $\delta_i = \delta(B^i)$ *denotes the* diameter *of the set* B^i.

PROOF. Let $\phi \in W^{2,\infty}$ be an arbitrary test-function. Using Taylor expansion around the point $x = x^i$ yields

$$\phi(x) = \phi(x^i) + (x - x^i)^* \, D\phi(x^i)$$

$$+ (x - x^i)^* \left\{ \int_0^1 (1-u) \, D^2\phi[ux + (1-u)x^i] \, du \right\} (x - x^i) ,$$

and the definition (A.4) gives

$$E_i'(\phi) = \int_{B^i} (x - x^i)^* \left\{ \int_0^1 (1-u) \, D^2\phi[ux + (1-u)x^i] \, du \right\} (x - x^i) \, dx .$$

Therefore

$$|E_i'(\phi)| \leq \tfrac{1}{2} \, |\phi|_{2,\infty,\widehat{B}^i} \int_{B^i} \|x - x^i\|^2 \, \mu(dx) \leq \tfrac{1}{2} \, |\phi|_{2,\infty,\widehat{B}^i} \, \delta_i^2 \, a^i ,$$

where $\widehat{B}^i$ denotes the *convex hull* of B^i. Then

$$|\langle \mu, \phi \rangle - \langle \mu^h, \phi \rangle| = |E'(\phi)| \leq \tfrac{1}{2} \sum_{i \in I} |\phi|_{2,\infty,\widehat{B}^i} \, \delta_i^2 \, a^i , \tag{A.6}$$

and

$$\|\mu - \mu^h\|_{-2,1} = \sup_{\phi \in W^{2,\infty}} \frac{|\langle \mu, \phi \rangle - \langle \mu^h, \phi \rangle|}{\|\phi\|_{2,\infty}} \leq \tfrac{1}{2} \sum_{i \in I} \delta_i^2 \, a^i . \qquad \square$$

Remark A.4 If the partition $\{B_i \, , \, i \in I\}$ is *given*, with $\delta_i \leq C \, h$ for all $i \in I$, then

$$\|\mu - \mu^h\|_{-2,1} \leq C \, h^2 .$$

On the other hand, if the partition $\{B_i \, , \, i \in I\}$ has to be chosen so as to make the quadrature error as small as possible, then estimate (A.5) can be used to derive the following criterion

$$\delta_i^2 \, a^i = c \qquad \text{for all } i \in I .$$

This criterion based on *equidistribution* of the local quadrature error, has the following interesting property

- a set with a large mass, will be split into some smaller subsets,
- conversely, neighbouring sets with small masses, will be packed together into one single set.

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Numerical Approximation for Nonlinear Filtering and Finite–Time Observers

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I Models of Partially Observed Systems

We consider partially observed systems of the form

$$\begin{cases} dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t \\ dY_t = h(X_t)\,dt + dV_t \end{cases} \tag{1}$$

where $\{W_t\,,\,t \geq 0\}$ and $\{V_t\,,\,t \geq 0\}$ are independent Wiener processes of appropriate dimensions, with covariance matrices I and R respectively. We are interested in the state estimation problem, under various hypotheses concerning $a = \sigma\,\sigma^*$ and R.

Consider first the two extreme cases : If $a \equiv 0$ and $R \equiv 0$, we are dealing with an *observer* problem for the deterministic system

$$\begin{cases} \dot{X}_t = b(X_t) \\ z_t = h(X_t) \end{cases} \tag{2}$$

At the other extreme, if R is non–singular, we are dealing with a *nonlinear filtering* problem for the diffusion process (1). We can also easily handle the following intermediate case : If R is non–singular and $a \equiv 0$, we are again dealing with a *nonlinear filtering* problem but the state equation is now an ODE

$$\begin{cases} \dot{X}_t = b(X_t) \\ dY_t = h(X_t)\,dt + dV_t \end{cases}$$

Let us point out that the solution of the state estimation problem is radically different, depending on whether R is non–singular or identically zero. On the other hand, whether a is non–singular, singular or identically zero only affects the algorithms to be used.

Our purpose is to present, for each of the three main cases described above, a solution to the state estimation problem, and to suggest some numerical approximation procedures. The general idea is to study the asymptotics $R \to 0$. As a by-product, we expect to obtain some numerical algorithms for the nonlinear filtering problem, that are robust when the non–singular matrix R is *small*.

II Solutions to State Estimation Problems

We assume for simplicity that $b \in C_b^1(\mathbf{R}^m, \mathbf{R}^m)$ and $h \in C_b^1(\mathbf{R}^m, \mathbf{R}^p)$, unless otherwise stated.

Let us begin with the *nonlinear filtering* (NLF) problem.

□ When R is non–singular, the Bayesian approach to the state estimation problem is to compute the unnormalized conditional probability distribution $\mu_t(dx)$ of the state X_t, given the past observations $\mathcal{Y}_t = \sigma(Y_s\,, 0 \leq s \leq t)$. By definition

$$\langle \mu_t, f \rangle = \mathbf{E}^\dagger[f(X_t)\, Z_t \mid \mathcal{Y}_t]\,,$$

for any test function f, where

$$Z_t^s = \exp\left\{\int_s^t h^*(X_\tau) R^{-1}\, dY_\tau - \tfrac{1}{2}\int_s^t |h(X_\tau)|^2_{R^{-1}}\, d\tau\right\} \quad \text{and} \quad Z_t = Z_t^0\,,$$

and $P^\dagger$ is the *reference* probability measure. The probability distribution $\mu_t(dx)$ satisfies a stochastic PDE in weak sense. Usually, this p.d.f. has a density w.r.t. the Lebesgue measure, i.e. $\mu_t(dx) = p_t(x)\, dx$. A sufficient condition for this to hold, is that the probability distribution $\mu_0(dx)$ of the initial condition X_0 has already a density w.r.t. the Lebesgue measure, i.e. $\mu_0(dx) = p_0(x)\, dx$. We will assume that $p_0(x) > 0$ for all $x \in \mathbf{R}^m$. The unnormalized conditional density $p_t(x)$ is the unique solution of the Zakai equation

$$dp_t = L^* p_t\, dt + p_t h^* R^{-1}\, dY_t\,, \tag{3}$$

with initial condition $p_0(x)$, where L^* is the formal adjoint of the second–order partial differential operator

$$L = \tfrac{1}{2}\,\mathrm{tr}[\, a\, \frac{\partial^2}{\partial x^2}\,] + b \cdot \frac{\partial}{\partial x}\,,$$

associated with the SDE

$$dX_t = b(X_t)\, dt + \sigma(X_t)\, dW_t\,. \tag{4}$$

□ If in addition $\sigma \equiv 0$, then the Zakai equation (3) becomes a first order stochastic PDE, for which a representation result is available in terms of the flow $\Phi_t(x)$ associated with the ODE

$$\dot{X}_t = b(X_t)\,. \tag{5}$$

Actually, define

$$J_{t-s}(x) = \det[\, \frac{\partial \Phi_{t-s}}{\partial x}(x)\,] = \exp\left\{\int_s^t \mathrm{div}\; b(\Phi_{\tau-s}(x))\, d\tau\right\}\,,$$

$$\Xi_{s,t}(x) = \exp\left\{\int_s^t h^*(\Phi_{\tau-s}(x)) R^{-1}\, dY_\tau - \tfrac{1}{2}\int_s^t |h(\Phi_{\tau-s}(x))|^2_{R^{-1}}\, d\tau\right\}\,.$$

In this case, the unique solution of the Zakai equation (3) satisfies

$$p_t(\Phi_{t-s}(x)) \cdot J_{t-s}(x) = \Xi_{s,t}(x) \cdot p_s(x)\,, \tag{6}$$

or equivalently, introducing the logarithmic transform $W_t(x) = -\log p_t(x)$

$$W_t(\Phi_{t-s}(x)) - \log J_{t-s}(x) = W_s(x) - \log \Xi_{s,t}(x)\,. \tag{7}$$

We turn now to the *observer* problem.

Let $\{x_t^*, 0 \le t \le T\}$ denote the *true* state trajectory producing the available observation trajectory $\{z_t, 0 \le t \le T\}$. The idea is to build an observer by considering the limit of a sequence of nonlinear filtering problems with noise covariances going to zero. Two different cases are possible

- Introduce small noises of similar intensities in both the state equation and the observation, i.e. set $a = \varepsilon I$ and $R = \varepsilon I$,
- Introduce a small noise in the observation only, i.e. set $a \equiv 0$ and $R = \varepsilon I$.

□ In the first case, it is proved in James [2] that

$$-\varepsilon \log p_t^\varepsilon(x) \xrightarrow{\varepsilon \downarrow 0} m_t'(x)$$

in probability uniformly on compact subsets of $x \in \mathbf{R}^m$, where up to an additive constant independent of x, $m_t'(x)$ is the unique solution of the Hamilton–Jacobi equation

$$\frac{\partial m_t'}{\partial t} + \frac{1}{2}\left|\frac{\partial m_t'}{\partial x}\right|^2 + b \cdot \frac{\partial m_t'}{\partial x} - V_t = 0 \ , \tag{8}$$

with initial condition $m_0'(x) = 0$, in the *viscosity* sense, where

$$V_t(x) = \tfrac{1}{2}|z_t - h(x)|^2 \ .$$

In addition, $m_t'(x)$ is the value function associated with the following control problem. Introduce first the action functional

$$I_t(\xi) = \tfrac{1}{2}\int_0^t |\dot{\xi}_s - b(\xi_s)|^2 \, ds$$

if $\xi \in C([0,T]\,;\mathbf{R}^m)$ is absolutely continuous, and $I_t(\xi) = +\infty$ otherwise. Define also

$$F_t(\xi) = \int_0^t V_s(\xi_s)\, ds = \tfrac{1}{2}\int_0^t |z_s - h(\xi_s)|^2 \, ds \ .$$

Then

$$m_t'(x) = \inf\left\{I_t(\xi) + F_t(\xi) \ : \ \xi_t = x\right\} \ .$$

Clearly $m_t'(x) \ge 0$ and $m_t'(x_t^*) = 0$ for the *true* state trajectory, and we define our observer as the set

$$\widehat{x}_t' = \underset{x \in \mathbf{R}^m}{\operatorname{argmin}} \ m_t'(x) = \left\{x \in \mathbf{R}^m \ : \ m_t'(x) = 0\right\} \ . \tag{9}$$

Obviously $x_t^* \in \widehat{x}_t'$ for all $t \ge 0$. It is proved in James [3] that, provided the deterministic system (2) is *observable* on $[0,T]$ (i.e. the map $x_0 \mapsto \{z_s, 0 \le s \le t\}$ is injective), the set-valued observer (9) is actually a ***finite-time observer*** (FTO) on $[0,T]$ (meaning that $\widehat{x}_t'$ is defined in terms of a recursive system with the property that $\widehat{x}_t' = \{x_t^*\}$ for all $t \ge T$).

□ In the second case, it follows from equation (7) that

$$-\varepsilon \log p_t^\varepsilon(x) = \varepsilon W_t^\varepsilon(x) \xrightarrow{\varepsilon\downarrow 0} m_t(x) \ ,$$

in probability uniformly on compact subsets of $x \in \mathbf{R}^m$, where up to an additive constant independent of x, $m_t(x)$ is given by

$$m_t(\Phi_t(x)) = \int_0^t V_s(\Phi_s(x))\,ds \quad \text{or} \quad m_t(x) = \int_0^t V_s(\Phi_{t-s}^{-1}(x))\,ds \ ,$$

i.e. $m_t(x) = F_t(\xi^{t,x})$, where $\xi^{t,x}$ is the unique solution of the ODE (5) ending in x at time t. In addition, $m_t(x)$ is the unique solution of the linear first-order PDE

$$\frac{\partial m_t}{\partial t} + b \cdot \frac{\partial m_t}{\partial x} - V_t = 0 \ ,$$

satisfying the initial condition $m_0 = 0$. Just as above, it is clear that $m_t(x) \geq 0$ and $m_t(x_t^*) = 0$ for the *true* state trajectory, and we define our observer as the set

$$\widehat{x}_t = \operatorname*{argmin}_{x \in \mathbf{R}^m} \ m_t(x) = \Big\{x \in \mathbf{R}^m \ : \ m_t(x) = 0\Big\} \ . \tag{10}$$

Here again, it is obvious that $x_t^* \in \widehat{x}_t$ for all $t \geq 0$, and in addition the set-valued observer defined by (10) is actually a FTO on $[0, T]$, provided the deterministic system (2) is *observable* on $[0, T]$.

Note that $m_t(x) = F_t(\xi^{t,x})$ where $I_t(\xi^{t,x}) = 0$ (i.e. $\xi^{t,x}$ solves the ODE (5) exactly) and $\xi_t^{t,x} = x$, whereas in the definition of $m_t'(x)$, a penalty $I_t(\xi)$ is put on those trajectories ξ that do not solve the ODE (5). This is a less severe requirement, and is reflected in the relation $m_t'(x) \leq m_t(x)$. Note however that $\widehat{x}_t = \widehat{x}_t'$. This is the set of those points that are *indistinguishable* from the true state x_t^*. In conclusion, the observer (10) is more *precise* than the observer (9), whereas the latter is expected to be more *robust* w.r.t. modeling errors.

III Numerical Approximation

In this section, we restrict ourselves to the situation where the state satisfies an ODE, in which case the solution to the NLF problem is given by (6), where R is non-singular, and the corresponding FTO is given by (10), where $R \equiv 0$.

Concerning the approximation of the NLF (6), we wish to compute an approximate normalized conditional density $p_k^{\Delta,\delta}(x)$ (where Δ and δ denote the time discretization step and the space discretization step respectively) with the following property

($\star$) as $\Delta, \delta \downarrow 0$

$$\mathbf{E} \int_{\mathbf{R}^m} |p_{[t/\Delta]}^{\Delta,\delta}(x) - c_t p_t(x)|\,dx \longrightarrow 0 \quad \text{for all } t \geq 0 \ ,$$

where c_t is a normalization constant.

Concerning the approximation of the FTO (10), our approach is to build a family $\widehat{x}_k^{\Delta,\delta}$ with the following property

($\star\star$) if the deterministic system (2) is observable on $[0,T]$, then as $\Delta, \delta \downarrow 0$

$$\mathrm{dist}(\widehat{x}^{\Delta,\delta}_{[t/\Delta]}, \{x^*_t\}) \longrightarrow 0 \quad \text{for all } t \geq T\ .$$

A necessary and sufficient condition for ($\star\star$) to hold is $\mathrm{dist}(\widehat{x}^{\Delta,\delta}_{[t/\Delta]}, \widehat{x}_t) \to 0$ as $\Delta, \delta \downarrow 0$. The approximate observer $\widehat{x}^{\Delta,\delta}_k$ will be defined in terms of an approximate value function $m^{\Delta,\delta}_k(x)$, i.e.

$$\widehat{x}^{\Delta,\delta}_k = \left\{ x \in \mathbf{R}^m \, : \, m^{\Delta,\delta}_k(x) \leq c^{\Delta,\delta} \right\}\ ,$$

and a sufficient condition for ($\star\star$) to hold is $c^{\Delta,\delta} \downarrow 0$ and $m^{\Delta,\delta}_{[t/\Delta]}(x) \to m_t(x)$ uniformly on compact subsets of $\mathbf{R}^m$, as $\Delta, \delta \downarrow 0$.

Time Discretization

Consider a uniform partition $0 = t_0 < \cdots < t_k < \cdots$ of the time interval $[0,\infty)$, with time step $\Delta = t_k - t_{k-1}$. The first step is to sample the available observation trajectory.

The nonlinear filtering problem. If noisy observations $\{Y_t\, , \, t \geq 0\}$ are available, we first build the following sequence of compressed observations

$$y^\Delta_k = \frac{1}{\Delta}[Y_{t_k} - Y_{t_{k-1}}] = \frac{1}{\Delta}\int_{t_{k-1}}^{t_k} h(X_s)\, ds + \frac{1}{\Delta}[V_{t_k} - V_{t_{k-1}}]$$

and we use the approximate model

$$\left\{ \begin{aligned} \dot{X}_t &= b(X_t) \\ y^\Delta_k &= h(X_{t_k}) + v^\Delta_k \end{aligned} \right. \tag{11}$$

where $\{v^\Delta_k\, , \, k = 1, 2, \cdots\}$ is a Gaussian white noise sequence with covariance matrix R/Δ.

The solution of the NLF problem for the approximate model (11) is given in terms of the *a priori* and *a posteriori* conditional probability densities defined by

$$p^\Delta_{k-\frac{1}{2}}(x)\, dx = P(X_{t_k} \in dx \mid \mathcal{Y}^\Delta_{k-1}) \quad \text{and} \quad p^\Delta_k(x)\, dx = P(X_{t_k} \in dx \mid \mathcal{Y}^\Delta_k)\ ,$$

respectively, where $\mathcal{Y}^\Delta_k = \sigma(y^\Delta_1, \cdots, y^\Delta_k)$. The transition from $p^\Delta_{k-1}(x)$ to $p^\Delta_k(x)$ is divided into two steps

- *prediction step* : Transport by the flow gives $p^\Delta_{k-\frac{1}{2}}(x) = T_\Delta\, p^\Delta_{k-1}(x)$ where $\{T_t\, , \, t \geq 0\}$ is the semigroup associated with the linear first-order PDE

$$\frac{\partial p_t}{\partial t} = L^* p_t\ . \tag{12}$$

An explicit solution is available for this equation

$$p^\Delta_{k-\frac{1}{2}}(\Phi_\Delta(x)) \cdot J_\Delta(x) = p^\Delta_{k-1}(x)\ , \tag{13}$$

or equivalently

$$\int_A p^\Delta_{k-\frac{1}{2}}(x)\, dx = \int_{\Phi_\Delta^{-1}(A)} p^\Delta_{k-1}(x)\, dx\ , \tag{14}$$

for all Borel set $A \subset \mathbf{R}^m$.

· *correction step* : According to the Bayes formula

$$p_k^\Delta(x) = c_k \cdot \Psi_k^\Delta(x) \cdot p_{k-\frac{1}{2}}^\Delta(x) , \tag{15}$$

where

$$\Psi_k^\Delta(x) = \exp\left\{-\tfrac{1}{2}\Delta \, |y_k^\Delta - h(x)|_{R^{-1}}^2\right\} ,$$

is the *likelihood function* for the estimation of X_{t_k} in the approximate model (11), based on the observation y_k^Δ alone, and c_k is a normalization constant.

Introducing the logarithmic transform $W_k^\Delta(x) = -\log p_k^\Delta(x)$, it follows from from (13) and (15) that

$$W_k^\Delta(x) - \log J_\Delta(\Phi_\Delta^{-1}(x)) = -\log c_k + W_{k-1}^\Delta(\Phi_\Delta^{-1}(x)) + \tfrac{1}{2}\Delta \, |y_k^\Delta - h(x)|_{R^{-1}}^2 . \tag{16}$$

The observer problem. If perfect observations $\{z_t \, , \, t \geq 0\}$ are available, i.e. $R \equiv 0$, we can simply use $z_k = z_{t_k}$, and our model becomes

$$\left\{ \begin{aligned} \dot{X}_t &= b(X_t) \\ z_k &= h(X_{t_k}) \end{aligned} \right. \tag{17}$$

Introducing the asymptotics $R = \varepsilon I$ in the NLF problem and sending ε to zero, it follows from equation (16) that

$$-\varepsilon \log p_k^{\Delta,\varepsilon}(x) = \varepsilon W_k^{\Delta,\varepsilon}(x) \xrightarrow{\varepsilon\downarrow 0} m_k^\Delta(x) ,$$

in probability uniformly on compact subsets of $x \in \mathbf{R}^m$, where $m_k^\Delta(x)$ satisfies the following relation

$$m_k^\Delta(x) = m_{k-1}^\Delta(\Phi_\Delta^{-1}(x)) + \Delta \; V_k^\Delta(x) ,$$

where

$$V_k^\Delta(x) = \tfrac{1}{2}|z_k^\Delta - h(x)|^2 \quad \text{and} \quad z_k^\Delta = \frac{1}{\Delta}\int_{t_{k-1}}^{t_k} z_s \, ds = \frac{1}{\Delta}\int_{t_{k-1}}^{t_k} h(X_s) \, ds \neq z_k .$$

It is clear that $m_k^\Delta(x) \geq 0$. However, because the averaged observation z_k^Δ used in the definition of $m_k^\Delta(x)$ is different from the actual observation z_k, we have $V_k^\Delta(x_{t_k}^*) \neq 0$ in general for the *true* state trajectory. Therefore, we decide to use the actual observation z_k in the definition of $m_k^\Delta(x)$, instead of the averaged observation z_k^Δ, i.e.

$$m_k^\Delta(x) = m_{k-1}^\Delta(\Phi_\Delta^{-1}(x)) + \Delta \; V_k(x) , \tag{18}$$

where

$$V_k(x) = \tfrac{1}{2}|z_k - h(x)|^2 .$$

This relation can be divided into two steps

· *prediction step* : Transport by the flow gives $m_{k-\frac{1}{2}}^\Delta(x) = S_\Delta \; m_{k-1}^\Delta(x)$ where $\{S_t \, , \, t \geq 0\}$ is the semigroup associated with the linear first-order PDE

$$\frac{\partial m_t}{\partial t} + b \cdot \frac{\partial m_t}{\partial x} = 0 . \tag{19}$$

An explicit solution is available for this equation

$$m_{k-\frac{1}{2}}^\Delta(\Phi_\Delta(x)) = m_{k-1}^\Delta(x) . \tag{20}$$

· *correction step* : The contribution of the new observation z_k to the approximate value function is given by

$$m_k^\Delta(x) = m_{k-\frac{1}{2}}^\Delta(x) + \Delta\ V_k(x)\ .$$

We note that

$$m_{k-\frac{1}{2}}^\Delta(x) = F_{k-1}^\Delta(\xi^{t_k,x}) \quad \text{and} \quad m_k^\Delta(x) = F_k^\Delta(\xi^{t_k,x})\ ,$$

where $\xi^{t,x}$ is the unique solution of the ODE (5) ending in x at time t, and the functional $F_k^\Delta(\xi)$ satisfies for all $\xi \in C([0,T]\,;\mathbf{R}^m)$

$$F_k^\Delta(\xi) = F_{k-1}^\Delta(\xi) + \Delta\ V_k(\xi_{t_k}) = \Delta\ \left\{V_1(\xi_{t_1}) + \cdots + V_k(\xi_{t_k})\right\}\ .$$

Now it is clear that $m_k^\Delta(x) \geq 0$ and $m_k^\Delta(x_{t_k}^*) = 0$ for the *true* state trajectory, and we define our observer as the set

$$\hat{x}_k^\Delta = \underset{x\in\mathbf{R}^m}{\operatorname{argmin}}\ m_k^\Delta(x) = \left\{x \in \mathbf{R}^m\ :\ m_k^\Delta(x) = 0\right\}\ . \tag{21}$$

Obviously $x_{t_k}^* \in \hat{x}_k^\Delta$ for all k, and one can verify using the explicit formulas that $m_{[t/\Delta]}^\Delta(x) \to m_t(x)$ uniformly on compact subsets as $\Delta \downarrow 0$, with the consequence that property $(\star\star)$ holds for this discrete–time approximation.

Model Approximation and PDE Discretization

To obtain computable algorithms, it is necessary to discretize the linear first–order PDE (12) or (19) involved in the prediction step. Generally speaking, two classes of methods can be used : in the *finite difference* approximation (FD) a fixed bounded grid is used, and partial differential operators are approximated by finite differences on grid points, whereas in the *flow-based* approximation (FLOW) the explicit representation (13) or (20) is used to move grid points (or alternatively cells) along the flow of the ODE (5).

A Finite Difference

A finite difference nonlinear filter. To derive a finite difference algorithm, we must first constrain the nonlinear filtering problem to a bounded domain. Let $D \subset \mathbf{R}^m$ be a m–dimensional open cube. After Dupuis-Ishii [1], we constrain the ODE (5) to the convex set $\bar{D}$ as follows. For $x \in \partial D$, let $\nu(x) = \{\ \nu \in \mathbf{R}^m\ :\ |\nu| = 1,\ \langle \nu, x - z\rangle \leq 0 \text{ for all } z \in \bar{D}\ \}$ denote the set of inward unit normals. For $x \in \bar{D}$, $v \in \mathbf{R}^m$, the projection $\pi(x,v)$ of the velocity vector v at x is given by v if $x \in D$, or $v + [\langle v, -\nu^*(x,v)\rangle \vee 0]\ \nu^*(x,v)$ if $x \in \partial D$, where $\nu^*(x,v)$ is an element of $\nu(x)$ which maximizes $\langle v, -\nu\rangle$, $\nu \in \nu(x)$. Define then $\tilde{b}(x) = \pi(x, b(x))$, $x \in \bar{D}$. By Theorem 5.1 of Dupuis-Ishii [1], there exists a unique absolutely continuous solution of the constrained ODE

$$\dot{\xi}_s = \tilde{b}(\xi_s) \quad \text{a.e. } 0 \leq s \leq t \tag{22}$$

satisfying $\xi_0 = x \in \bar{D}$.

A finite difference algorithm is obtained using a Markov chain scheme similar to those described in Kushner [5]. Let $\mathbf{R}_\delta^m$ denote a coordinate grid of size $\delta > 0$. We define a system of neighborhoods

$N_\delta(x) = \{\ z \in \mathbf{R}^m_\delta\ :\ z = x$ or $z = x \pm \delta e_i$ for some $i = 1, \ldots, m\ \}$ for $x \in \mathbf{R}^m_\delta$, where $e_i \in \mathbf{R}^m$ denotes the i-th unit vector. We define next $\bar{D}^\delta = \bar{D} \cap \mathbf{R}^m_\delta$, $D^\delta = \{\ x \in \bar{D}^\delta\ :\ N_\delta(x) \subset \bar{D}^\delta\ \}$, and $\partial D^\delta = \bar{D}^\delta \setminus D^\delta$. We define the jump intensity matrix $L_\delta(x,z)$ of a pure jump Markov process $\{\overline{X}^\delta_t,\ t \geq 0\}$ taking values in $\bar{D}^\delta$ by

$$L_\delta(x,z) = \begin{cases} -|\tilde{b}(x)|_1/\delta & \text{if } z = x, \\ \tilde{b}^\pm_i(x)/\delta & \text{if } z = x \pm \delta e_i \text{ and } i = 1, \ldots, m \\ 0 & \text{if } z \notin N_\delta(x), \end{cases} \tag{23}$$

with the notation $|u|_1 = |u_1| + \cdots + |u_m|$ for any $u = (u_1, \cdots, u_m)$. If we use an *implicit* time discretization scheme, we obtain the finite difference equation

$$p^{\Delta,\delta}_k(x) - \Delta \sum_{z \in N_\delta(x)} L^*_\delta(x,z)\, p^{\Delta,\delta}_k(z) = c_k \cdot \Psi^\Delta_k(x) \cdot p^{\Delta,\delta}_{k-1}(x)\ , \tag{24}$$

for $x \in \bar{D}^\delta$ and $k = 1, \cdots$, where c_k is a normalization constant, and the initial condition $p^{\Delta,\delta}_0(x)$ is a suitable approximation of the density $p_0(x)$. This relation can be divided into two steps

· *prediction step* : Transport by the flow gives

$$[I - \Delta\ L^*_\delta]\ p^{\Delta,\delta}_{k-\frac{1}{2}}(x) = p^{\Delta,\delta}_{k-1}(x)\ .$$

· *correction step* : According to the Bayes formula

$$p^{\Delta,\delta}_k(x) = c_k \cdot \Psi^\Delta_k(x) \cdot p^{\Delta,\delta}_{k-\frac{1}{2}}(x)\ ,$$

where c_k is a normalization constant.

The following result is proved in Kushner [5] using weak convergence $\overline{X}^\delta_\cdot \Longrightarrow X_\cdot$ as $\delta \downarrow 0$.

Theorem 1 *Property* $(\star)$ *holds for the finite difference nonlinear filtering algorithm.*

A finite difference observer. To derive a finite difference algorithm, we still need to constrain the observer problem to a bounded domain. However, because we are going to approximate (18), we must consider the ODE (5) as running backward in time, before we constrain it to the convex set $\bar{D}$. We use the same definition as above for the set $\nu(x)$ of inward unit normals. For $x \in \bar{D}$, $v \in \mathbf{R}^m$, the projection $\pi(x,v)$ of the velocity vector v at x is now given by v if $x \in D$, or $v + [\langle v, \nu^*(x,v)\rangle \vee 0]\ \nu^*(x,v)$ if $x \in \partial D$, where $\nu^*(x,v)$ is an element of $\nu(x)$ which maximizes $\langle v, \nu\rangle$, $\nu \in \nu(x)$. Define then $\tilde{b}(x) = -\pi(x, -b(x))$, $x \in \bar{D}$. By Theorem 5.1 of Dupuis-Ishii [1] again, there exists a unique absolutely continuous solution $\xi = \xi^{x,t}$ of the constrained ODE

$$\dot{\xi}_s = \tilde{b}(\xi_s) \quad \text{a.e. } 0 \leq s \leq t\ , \tag{25}$$

satisfying $\xi_t = x \in \bar{D}$.

Select $\beta \in C(\mathbf{R}^m)$ non-negative, $\beta \equiv 0$ on $D' \subset D$, with $D' \cap \partial D = \emptyset$, and $\beta > 0$ on ∂D. Now define the value function for $x \in \bar{D}$, $t \geq 0$ by

$$m_t(x) = \beta(\xi_0) + \int_0^t [V_s(\xi_s) + \beta(\xi_s)]\, ds \ , \tag{26}$$

where ξ is the solution of the constrained ODE (25). Then $m_t(x)$ is the unique viscosity solution of the Hamilton–Jacobi equation, see Lions [6]

$$\begin{cases} \dfrac{\partial m_t}{\partial t} + b \cdot \dfrac{\partial m_t}{\partial x} - V_t - \beta = 0 & \text{in } D \times (0, S] \ , \\[2ex] \qquad\qquad -\nu \cdot \dfrac{\partial m_t}{\partial x} = 0 & \text{on } \partial D \times (0, S] \ , \end{cases} \tag{27}$$

satisfying the initial condition $m_0(x) = \beta(x)$ for $x \in D$. In addition, m satisfies in the viscosity sense

$$\frac{\partial m_t}{\partial t} + \tilde{b} \cdot \frac{\partial m_t}{\partial x} - V_t - \beta = 0 \quad \text{on } \partial D \times (0, S] \ . \tag{28}$$

Define the corresponding observer as the set

$$\hat{x}_t = \operatorname*{argmin}_{x \in \mathbf{R}^m} \ m_t(x) = \left\{ x \in \mathbf{R}^m \ : \ m_t(x) = 0 \right\} \ . \tag{29}$$

Let $\mathcal{I} = \{x_0 \in D' \ : \ \Phi_s(x_0) \in D', \ 0 \leq s \leq S\}$. If $x_0^* \in \mathcal{I}$, then $x_t^* \in \hat{x}_t$ for all $0 \leq t \leq S$, and the observer (29) defines a FTO provided the deterministic system (2) is *observable* on $[0, T]$, see James [4].

We again use a Markov chain finite difference scheme. However, we discretize the boundary equation (28) rather than the boundary condition in (27). We use the same definition as above for the grid $\mathbf{R}_\delta^m$, the system of neighborhoods $N_\delta(x)$, and the subsets $\bar{D}^\delta$, D^δ and $\partial D^\delta = \bar{D}^\delta \setminus D^\delta$ of the grid $\mathbf{R}_\delta^m$. Assume that $v = \delta/\Delta$ is a fixed real number, indicating the "speed" of the algorithm, satisfying

$$v \geq \max_{x \in \bar{D}} |\tilde{b}(x)|_1 \ . \tag{30}$$

We define the transition probabilities $\pi^{\Delta,\delta}(x,z) = P(\xi_{k-1}^{\Delta,\delta} = z \mid \xi_k^{\Delta,\delta} = x)$ for a *backward* Markov chain $\{\xi_k^{\Delta,\delta}, \ k = [S/\Delta], \ldots, 0\}$ by

$$\pi^{\Delta,\delta}(x,z) \ = \ \begin{cases} 1 - |\tilde{b}(x)|_1/v & \text{if } z = x, \\[1ex] \tilde{b}_i^{\mp}(x)/v & \text{if } z = x \pm \delta e_i \text{ and } i = 1, \ldots, m \\[1ex] 0 & \text{if } z \notin N_\delta(x) \ . \end{cases} \tag{31}$$

Note that $\mathbf{E}[\xi_{k-1}^{\Delta,\delta} - \xi_k^{\Delta,\delta} \mid \xi_k^{\Delta,\delta} = x] = -\Delta\, \tilde{b}(x)$.

If we replace $\Phi_\Delta^{-1}(x)$ in (18) by the state $\xi_{k-1}^{\Delta,\delta}$ of the backward Markov chain starting at $\xi_k^{\Delta,\delta} = x$ and take expectations, we obtain the finite difference equation

$$m_k^{\Delta,\delta}(x) = \sum_{z \in N_\delta(x)} \pi^{\Delta,\delta}(x,z)\, m_{k-1}^{\Delta,\delta}(z) + \Delta\, [V_k(x) + \beta(x)] \ , \tag{32}$$

for $x \in \bar{D}^\delta$ and $k = 1, \ldots, [S/\Delta]$, with initial condition $m_0^{\Delta,\delta}(x) = \beta(x)$. This relation can be divided into two steps

· *prediction step* : Transport by the flow gives

$$m^{\Delta,\delta}_{k-\frac{1}{2}}(x) = \pi^{\Delta,\delta}\, m^{\Delta,\delta}_{k-1}(x) \ .$$

· *correction step* : The contribution of the new observation z_k to the approximate value function is given by

$$m^{\Delta,\delta}_{k}(x) = m^{\Delta,\delta}_{k-\frac{1}{2}}(x) + \Delta\, [V_k(x) + \beta(x)] \ .$$

The finite difference observer set is defined by

$$\widehat{x}^{\Delta,\delta}_k = \operatorname*{argmin}_{x \in \bar{D}^\delta}\ m^{\Delta,\delta}_k(x) \ . \tag{33}$$

Obviously, there is no reason for this approximate observer to satisfy the non–asymptotic consistency property : in general we can not guarantee that $x^*_{t_k} \in \widehat{x}^{\Delta,\delta}_k$.

The following result is proved in James [4].

Theorem 2 *If $x^*_0 \in \mathcal{I}$, then property $(\star\star)$ holds for the finite difference observer algorithm.*

Remark 3 It is also shown in [4] that under additional regularity assumptions $\mathrm{dist}(\widehat{x}^{\Delta,\delta}_{[t/\Delta]}, x^*_t) = O(\sqrt{\delta})$ as $\delta \downarrow 0$.

Remark 4 The speed constraint (30) which appears in the finite difference observer algorithm is actually a stability condition for the *explicit* time–discretization scheme used in (32). From the probabilistic point of view, it ensures that (31) defines transition probabilities. We do not need a similar constraint for the finite difference NLF algorithm, because we are using there an *implicit* time–discretization scheme.

B Flow–Based Approximation

Let us first describe the approximate model we are going to use.

We assume that at each time t_k, a partition $\{B^i_k\,,\, i \in I^\delta_k\}$ of the state space $\mathbf{R}^m$ is given, and we define the discrete I^δ_k–valued state n^δ_k by the relation

$$\left\{n^\delta_k = i\right\} = \left\{X_{t_k} \in B^i_k\right\} \ . \tag{34}$$

The idea behind our approximation is to suppose that, at each step of the algorithm, any information (e.g. probability distribution, likelihood function, value function) about the continuous state X_{t_k} is immediately compressed into some information about the discrete state n^δ_k. We can think of memory constraint as a justification for this compression mechanism. As a consequence, whenever information is needed about the continuous state X_{t_k}, it has to be deduced from the corresponding information about the discrete state n^δ_k, resulting in *compression error.*

Making explicit use of the flow associated with (5), we have

$$\left\{\xi_{t_k} \in B^i_k\right\} = \left\{\xi_{t_{k-1}} \in \Phi^{-1}_\Delta(B^i_k)\right\} = \bigcup_{j \in I^\delta_{k-1}[i]} \left\{\xi_{t_{k-1}} \in B^j_{k-1} \cap \Phi^{-1}_\Delta(B^i_k)\right\} \ , \tag{35}$$

where

$$I^{\delta}_{k-1}[i] = \left\{ j \in I^{\delta}_{k-1} \,:\, B^j_{k-1} \cap \Phi^{-1}_{\Delta}(B^i_k) \neq \emptyset \right\} ,$$

provided ξ solves the ODE (5). Notice that in general the set $I^{\delta}_{k-1}[i]$ has finite cardinality.

Various possible choices are available for the partitions, e.g.

- $I^{\delta}_k = I^{\delta}_{k-1} \equiv I^{\delta}$ and $B^i_k = \Phi_{\Delta}(B^i_{k-1})$ for all $i \in I^{\delta}$. In this case $n^{\delta}_k = n^{\delta}_{k-1}$, i.e. the discrete state process is constant over time, but the sets B^i_k can become very complicated after some steps.
- $I^{\delta}_k = I^{\delta}_{k-1} \equiv I^{\delta}$ and $B^i_k = B^i_{k-1}$ for all $i \in I^{\delta}$. In this case, the partition is constant over time, but updating the discrete state can be cumbersome.

Between these two extreme cases, a trade–off has to be found in order to reduce the computational burden of updating both the partition and the discrete probability distribution : B^i_k should both be "close" to $\Phi_{\Delta}(B^i_{k-1})$ and have a simple geometry.

A flow–based nonlinear filter. According to our approximation approach, we introduce the discrete *a priori* and *a posteriori* conditional probability distributions

$$\widetilde{\mu}^i_{k-\frac{1}{2}} = P(X_{t_k} \in B^i_k \mid \mathcal{Y}^{\Delta}_{k-1}) \quad \text{and} \quad \widetilde{\mu}^i_k = P(X_{t_k} \in B^i_k \mid \mathcal{Y}^{\Delta}_k) ,$$

respectively, where again $\mathcal{Y}^{\Delta}_k = \sigma(y^{\Delta}_1, \cdots, y^{\Delta}_k)$. Making use of (35) transport by the flow gives

$$\begin{aligned}
\widetilde{\mu}^i_{k-\frac{1}{2}} &= \sum_{j \in I^{\delta}_{k-1}[i]} P(X_{t_{k-1}} \in B^j_{k-1} \cap \Phi^{-1}_{\Delta}(B^i_k) \mid \mathcal{Y}^{\Delta}_{k-1}) \\
&= \sum_{j \in I^{\delta}_{k-1}[i]} \lambda(B^j_{k-1} \cap \Phi^{-1}_{\Delta}(B^i_k)) \cdot \frac{1}{\lambda(B^j_{k-1} \cap \Phi^{-1}_{\Delta}(B^i_k))} \int_{B^j_{k-1} \cap \Phi^{-1}_{\Delta}(B^i_k)} p^{\Delta}_{k-1}(x)\, dx \\
&\simeq \sum_{j \in I^{\delta}_{k-1}[i]} \lambda(B^j_{k-1} \cap \Phi^{-1}_{\Delta}(B^i_k)) \cdot \frac{1}{\lambda(B^j_{k-1})} \int_{B^j_{k-1}} p^{\Delta}_{k-1}(x)\, dx \\
&\simeq \sum_{j \in I^{\delta}_{k-1}[i]} \widetilde{\mu}^j_{k-1} \, \frac{\lambda(B^j_{k-1} \cap \Phi^{-1}_{\Delta}(B^i_k))}{\lambda(B^j_{k-1})} \, .
\end{aligned}$$

Next, according to the Bayes formula

$$\begin{aligned}
\widetilde{\mu}^i_k &= c_k \int_{B^i_k} \Psi^{\Delta}_k(x)\, p^{\Delta}_{k-\frac{1}{2}}(x)\, dx \\
&= c_k \int_{B^i_k} p^{\Delta}_{k-\frac{1}{2}}(x)\, dx \cdot \frac{\displaystyle\int_{B^i_k} \Psi^{\Delta}_k(x)\, p^{\Delta}_{k-\frac{1}{2}}(x)\, dx}{\displaystyle\int_{B^i_k} p^{\Delta}_{k-\frac{1}{2}}(x)\, dx} \\
&\simeq c_k \cdot \widetilde{\mu}^i_{k-\frac{1}{2}} \cdot \max_{x \in B^i_k} \Psi^{\Delta}_k(x) \, ,
\end{aligned}$$

where c_k is a normalization constant. This approximation can be justified in the small noise case, using the Laplace asymptotic formula.

To obtain a computable algorithm, we introduce new discrete probability distributions $\mu^i_{k-\frac{1}{2}}$ and μ^i_k, and the corresponding densities

$$p^{\Delta,\delta}_{k-\frac{1}{2}}(x) = \mu^i_{k-\frac{1}{2}}/\lambda(B^i_k) \qquad \text{and} \qquad p^{\Delta,\delta}_k(x) = \mu^i_k/\lambda(B^i_k) \qquad \text{iff } x \in B^i_k \; .$$

We then *define* the transition from $\{\mu^i_{k-1}\,,\, i \in I^\delta_{k-1}\}$ to $\{\mu^i_k\,,\, i \in I^\delta_k\}$, by the following two steps

· *prediction step* : Transport by the flow gives

$$\mu^i_{k-\frac{1}{2}} = \sum_{j \in I^\delta_{k-1}[i]} \mu^j_{k-1} \, \frac{\lambda(B^j_{k-1} \cap \Phi^{-1}_\Delta(B^i_k))}{\lambda(B^j_{k-1})} \; .$$

· *correction step* : The contribution of the new observation y^Δ_k is given by

$$\mu^i_k = c_k \cdot R^i_k \cdot \mu^i_{k-\frac{1}{2}} \; , \tag{36}$$

where c_k is a normalization constant, and

$$R^i_k = \max_{x \in B^i_k} \Psi^\Delta_k(x) \; ,$$

is the *generalized likelihood function* for the estimation of n^δ_k based on the observation y^Δ_k alone.

Theorem 5 *In the case* $I^\delta_k = I^\delta_{k-1} \equiv I^\delta$, *let* $\{B^i_0\,,\, i \in I^\delta\}$ *denote a finite partition of a bounded domain* D *with* $\mathrm{diam}(B^i_0) \leq \delta$. *Then property* $(\star)$ *holds for this flow-based nonlinear filtering algorithm.*

A flow-based observer. According to our approximation approach, we introduce the *a priori* and *a posteriori* discrete value functions

$$\widetilde{m}^i_{k-\frac{1}{2}} = \inf\left\{F^\Delta_{k-1}(\xi^{t_k,x}) \; : \; x \in B^i_k\right\} \quad \text{and} \quad \widetilde{m}^i_k = \inf\left\{F^\Delta_k(\xi^{t_k,x}) \; : \; x \in B^i_k\right\} \; , \tag{37}$$

respectively. Making use of (35) transport by the flow gives

$$\widetilde{m}^i_{k-\frac{1}{2}} = \inf_{j \in I^\delta_{k-1}[i]} \inf\left\{F^\Delta_{k-1}(\xi^{t_{k-1},x}) \; : \; x \in B^j_{k-1} \cap \Phi^{-1}_\Delta(B^i_k)\right\} \geq \inf_{j \in I^\delta_{k-1}[i]} \widetilde{m}^j_{k-1} \; .$$

Next, by definition of the functional $F^\Delta_k(\xi)$,

$$\widetilde{m}^i_k = \inf\left\{F^\Delta_{k-1}(\xi^{t_k,x}) + \Delta\, V_k(x) \; : \; x \in B^i_k\right\} \geq \widetilde{m}^i_{k-\frac{1}{2}} + \Delta \inf_{x \in B^i_k} V_k(x) \; .$$

Thus the discrete value functions satisfy difference *inequalities*. Unfortunately, this does not give a recursive mechanism for computation. Instead, we introduce new discrete value functions $m^i_{k-\frac{1}{2}}$ and m^i_k, and the corresponding value functions

$$m^{\Delta,\delta}_{k-\frac{1}{2}}(x) = m^i_{k-\frac{1}{2}} \qquad \text{and} \qquad m^{\Delta,\delta}_k(x) = m^i_k \qquad \text{iff } x \in B^i_k \; .$$

We then *define* the transition from $\{m^i_{k-1}\,,\, i \in I^\delta_{k-1}\}$ to $\{m^i_k\,,\, i \in I^\delta_k\}$ by the following two steps

· *prediction step* : Transport by the flow gives

$$m^i_{k-\frac{1}{2}} = \inf_{j \in I^\delta_{k-1}[i]} m^j_{k-1} .$$

· *correction step :* The contribution of the new observation z_k is given by

$$m^i_k = m^i_{k-\frac{1}{2}} + \Delta \inf_{x \in B^i_k} V_k(x) .$$

By construction, it is clear that $m^{\Delta,\delta}_k(x) \geq 0$ and $m^{\Delta,\delta}_k(x^*_{t_k}) = 0$ for the *true* state trajectory, and we define our observer as the set

$$\widehat{x}^{\Delta,\delta}_k = \operatorname*{argmin}_{x \in \mathbf{R}^m} m^{\Delta,\delta}_k(x) = \left\{ x \in \mathbf{R}^m \,:\, m^{\Delta,\delta}_k(x) = 0 \right\} , \tag{38}$$

or equivalently

$$\widehat{x}^{\Delta,\delta}_k = \bigcup_{i \in \widehat{I}^\delta_k} B^i_k \quad \text{with} \quad \widehat{I}^\delta_k = \left\{ i \in I^\delta_k \,:\, m^i_k = 0 \right\} .$$

By an inductive comparison argument, it is easy to show that $m^{\Delta,\delta}_k(x) \leq m^\Delta_k(x)$, with the consequence that $\widehat{x}^\Delta_k \subset \widehat{x}^{\Delta,\delta}_k$. Therefore, $x^*_{t_k} \in \widehat{x}^{\Delta,\delta}_k$.

Theorem 6 *In the case $I^\delta_k = I^\delta_{k-1} \equiv I^\delta$ and $B^i_k = \Phi_\Delta(B^i_{k-1})$ for all $i \in I^\delta$, let $\{B^i_0 \,,\, i \in I^\delta\}$ denote a finite partition of a bounded domain D with* $\operatorname{diam}(B^i_0) \leq \delta$*. If $x^*_0 \in D$, then property $(\star\star)$ will hold for this flow-based observer algorithm.*

As noticed in James [3], the only thing that matters is the argmin set, not the value function itself. This remark can be used to design a simplified algorithm for the construction of the set-valued observer (38). We introduce the piecewise-constant logical value functions $\overline{m}^{\Delta,\delta}_k(x)$ taking values TRUE or FALSE, and defined iteratively by the following relations

$$\overline{m}^i_{k-\frac{1}{2}} = \bigvee_{j \in I^\delta_{k-1}[i]} \overline{m}^j_{k-1} ,$$

$$\overline{m}^i_k = \overline{m}^i_{k-\frac{1}{2}} \wedge \overline{V}^i_k ,$$

where

$$\overline{V}^i_k = \begin{cases} \text{TRUE} & \text{if } \inf_{x \in B^i_k} V_k(x) = 0 \\ \\ \text{FALSE} & \text{otherwise} \end{cases}$$

It is clear that $\overline{m}^i_k = \text{TRUE}$ iff $m^i_k = 0$, so that an equivalent expression for the set-valued observer (38) is given by

$$\widehat{x}^{\Delta,\delta}_k = \bigcup_{i \in \widehat{I}^\delta_k} B^i_k \quad \text{with} \quad \widehat{I}^\delta_k = \left\{ i \in I^\delta_k \,:\, \overline{m}^i_k = \text{TRUE} \right\} .$$

Corollary 7 *Under the assumptions of Theorem 6, property $(\star\star)$ will hold for the simplified algorithm.*

Remark 8 In the particular case where $I_k^\delta = I_{k-1}^\delta \equiv I^\delta$ and $B_k^i = \Phi_\Delta(B_{k-1}^i)$ for all $i \in I^\delta$, the algorithms exhibit a parallel structure explicitly. On the other hand, these algorithms assume that certain calculations can be made exactly. This is not always possible, in which case one would have e.g. to discretize the ODE (5) or use the following approximations

$$\frac{1}{\lambda(B_k^i)} \int_{B_k^i} p(x)\, dx \simeq p(x_k^i) \qquad \max_{x \in B_k^i} \Psi_k^\Delta(x) \simeq \Psi_k^\Delta(x_k^i)\;,$$

$$\inf_{x \in B_k^i} m(x) \simeq m(x_k^i) \qquad \inf_{x \in B_k^i} V_k(x) \simeq V_k(x_k^i)\;,$$

where x_k^i is some point in B_k^i.

IV Numerical Experiments

A A One Dimensional Example

We consider a one dimensional model with

$$b(x,t) = -0.2x + 0.8\cos(2.5t) \qquad h(x) = \mathrm{sgn}(x) \ .$$

Even though the observation function is ***discontinuous***, the convergence results are still valid, see James [4]. The location of the trajectory is determined at the first time t^* it crosses the origin, so the system is observable.

Figure 1 (below) shows results for the simplified (logical) flow-based observer algorithm, with the choice $I_k^\delta = I_{k-1}^\delta \equiv I^\delta$ and $B_k^i = \Phi_\Delta(B_{k-1}^i)$ for all $i \in I^\delta$, $\Delta = 0.05$, $\delta = 0.02$, and noise-free observations. The estimate $\hat{x}_t$ is a one-dimensional set for times t before t^*, and zero-dimensional after this time.

Figure 2 illustrates the numerical results obtained from the finite difference nonlinear filter algorithm. Here, $\Delta = 0.05$, $\delta = 0.005$, $R = 10^{-4}$, and the observation path was noise-free. Notice the jumps in the conditional mean trajectory and the peaking of the conditional density function each time the origin is crossed. Numerical viscosity causes the density to spread between these times.

Figure 3 shows results for the finite difference observer algorithm, with $\delta = 0.02$, $\Delta = 0.0198$, $v = 1.01$, and noise-free observations. The plot of the value function clearly shows the valley containing the state trajectory.

Figure 4 shows results for the flow-based nonlinear filtering algorithm, with the choice $I_k^\delta = I_{k-1}^\delta \equiv I^\delta$ and $B_k^i = \Phi_\Delta(B_{k-1}^i)$ for all $i \in I^\delta$, $\Delta = 0.05$, $\delta = 0.02$, $R = 10^{-4}$, and noise-free observations. Marginals for the conditional density are shown for times before and after time t^*.

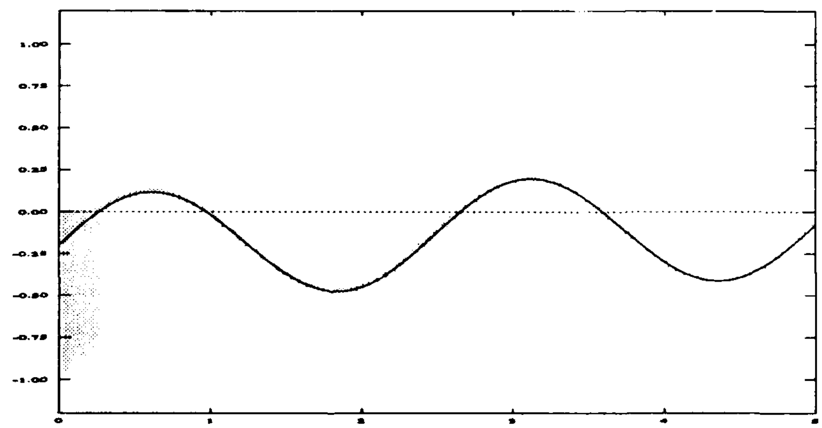


Figure 1. Flow-based observer, simplified algorithm.
State x_t and estimate $\hat{x}_t$ trajectories.

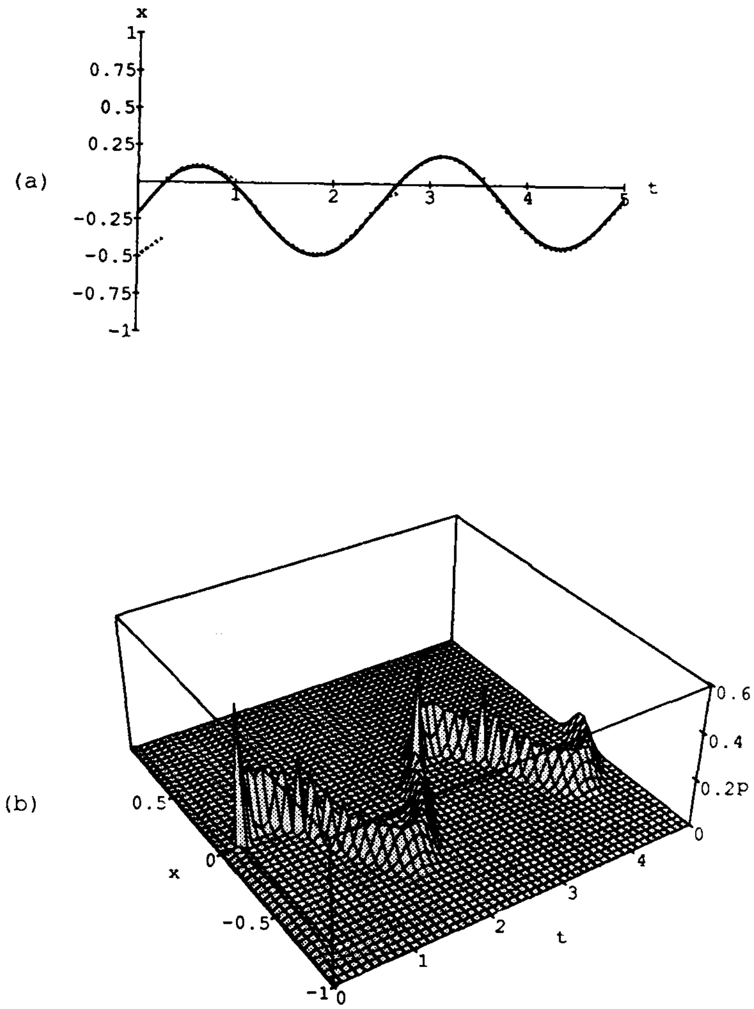


Figure 2. **Finite difference nonlinear filter.**
(a) State x_t and conditional mean $\mathbf{E}[x_t|\mathcal{Y}_t]$ trajectories; (b) Conditional density function.

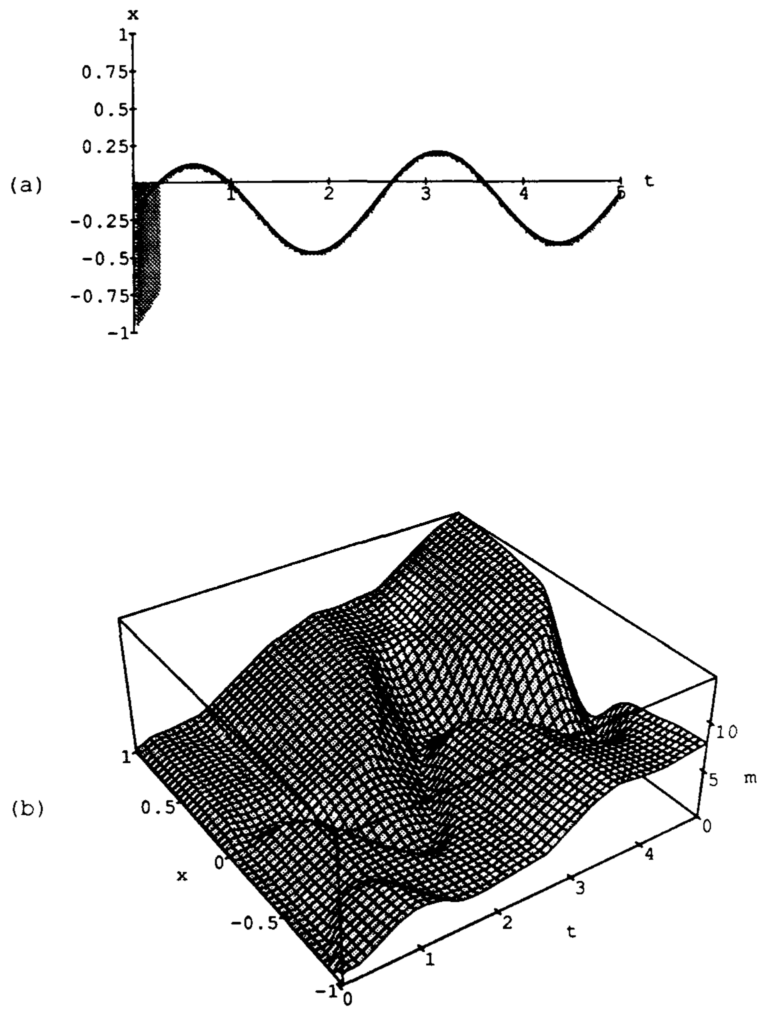
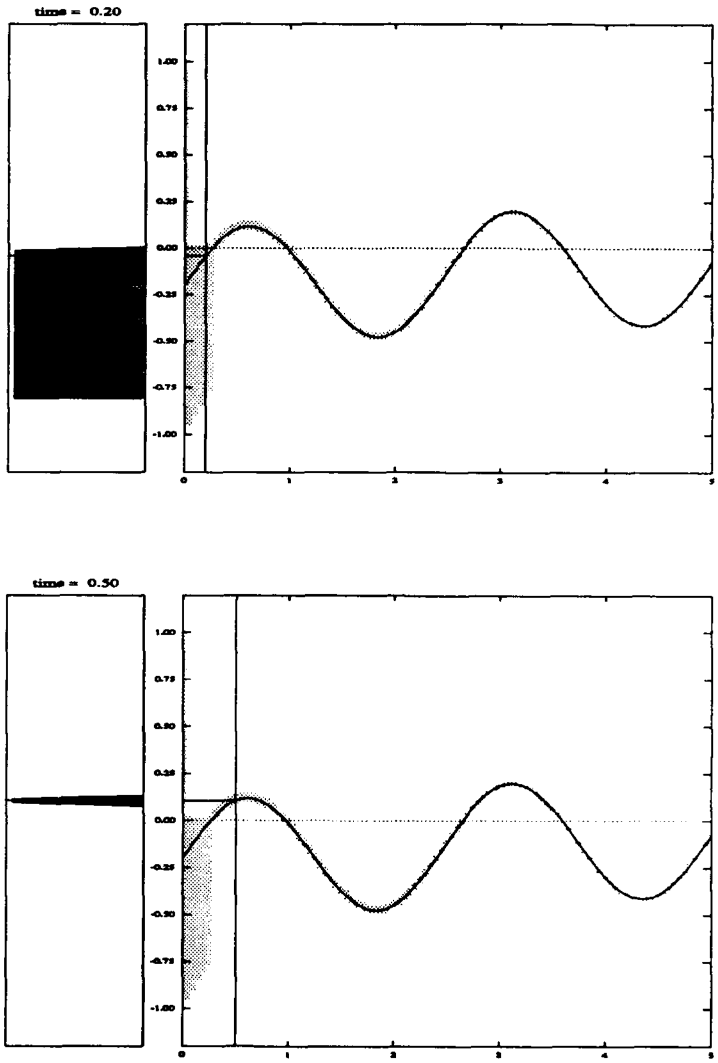


Figure 3. Finite difference observer.
(a) State x_t and estimate $\hat{x}_t$ trajectories; (b) Value function.



(a)

(b)

Figure 4. Flow–based nonlinear filter.
(a) State x_t trajectory, 90% confidence region, density at $t = 0.2$;
(b) State x_t trajectory, 90% confidence region, density at $t = 0.5$.

B A Four Dimensional Example

We consider here the problem of target motion analysis, which is to estimate the trajectory (position and velocity) of a target moving at constant speed along a straight line at the surface of the sea. We suppose that bearings–only measurements are available in discrete time, taken from a moving observation platform. If the observation platform itself moves at constant speed along a straight line, the problem is *non–observable*. However, as soon as the observation platform changes its course, the problem becomes *observable*. Assuming that the direction of motion of the target is known, which is true in the case of perfect observations, we can reduce the problem to three dimensions. The state vector is $X = (x, y, v)$ and the state equation

$$\dot{x}_t = v_t \qquad \dot{y}_t = 0 \qquad \dot{v}_t = 0 \ .$$

The observation function is

$$h(x, y, v, t) = \arctan[\frac{x - x_t^P}{y - y_t^P}] \ ,$$

where (x_t^P, y_t^P) is the (known) position of the observation platform at time t.

For this problem, the flow is known explicitly, and the flow–based algorithms (for both the nonlinear filtering and the observer case) are explicitly parallelizable. A variant of the flow–based NLF algorithm has been implemented at INRIA on a 16K Connection Machine from Thinking Machines Corporation. Numerical experiments have been carried out, using noisy observations with standard deviations ranging between one and five degrees. The goal is to find better maneuvers, and to investigate them off–line. The filter is not intended to be run in real–time on the ship.

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Finite Dimensional Approximate Filters in the case of High Signal–to–Noise Ratio

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Abstract

We present some recent results on nonlinear filtering problems with high signal–to–noise ratio. We concentrate mainly on the scalar case, where the observation function is not one to one. We describe two situations where a good suboptimal filter is provided by a bunch of one dimensional filters, together with statistical tests for choosing which filter should be followed.

1 Introduction

It is by now well known (see e.g. Pardoux [9]) that the nonlinear filtering problem is a difficult one, whose optimal solution is in most cases given only by the solution of an infinite dimensional equation, the Zakai equation.

On the other hand, up to now all pratical filtering problems are solved by approximate linear filters, in particular the well–known extended Kalman filter (see e.g. [9]). However, the extended Kalman filter does not rely on any mathematical foundation, it is known both from theory (see [9], Picard [14] and section 2 below) and experience that it sometimes behaves very poorly, while it gives very satisfactory results in many situations.

A good framework for a mathematical analysis of approximate filters, including the extended Kalman filter, is the situation of a

high signal–to–noise ratio, i.e. we consider the following non linear filtering problem :

$$\begin{cases} dX_t &= f(X_t)\,dt + g(X_t)dV_t \\ dY_t &= h(X_t)\,dt + \varepsilon dW_t \end{cases}$$

where $\{X_t\}$ is the unobserved process to be filtered, $\{Y_t\}$ is the observation process, $\{V_t\}$ and $\{W_t\}$ are mutually independent standard Wiener processes, all processes being scalar for simplicity. The goal is to obtain asymptotic results as $\varepsilon \to 0$ (with $\varepsilon > 0$).

In the case where h is one to one, this problem has first been analysed by Bobrovsky, Zakai [2] and Katzur, Bobrovsky, Schuss [6]. Jean Picard has then given a very complete mathematical analysis of this problem, see [10], [11], [12], [13], and Bensoussan [1] has given another proof of most of Picard's results. Those results can be very roughly summarized as follows : for small $\varepsilon > 0$, the variance of the conditional law is of order ε, the optimal and suboptimal filters have a short memory (old observations are quickly "forgotten" by the filter), and there exist various finite dimensional suboptimal filters whose output is close to the conditional expectation of X_t given $\{Y_s, 0 \le s \le t\}$ (including the extended Kalman filter and a one dimensional filter whose "error" is of the order of ε). Let us note that analogous results have been established for discrete–time problems by Milheiro [7].

A second class of problems concerns the case where h is *not* one to one. Two cases of such an h are as follows. Case A is where h is locally one to one ; in the situation $\dim X = \dim Y = 1$ which we shall consider below, this means that h is piecewise monotone. Case B is where $\dim X > \dim Y$ (say $\dim X = 2, \dim Y = 1$); and h is a function of say X_t^1 only, h being either monotone or piecewise monotone.

The aim of this paper is to present some recent results for the two above problems. We shall mainly be concerned with case A, and give some hints concerning case B.

The organisation of the paper is as follows. In section 2, we shall present case A, discuss the problem in case $\varepsilon = 0$ (no observation noise), introduce two "detectability assumptions" and compare them.

In section 3, we shall treat in some detail case A under one of the two detectability assumptions. In section 4, we shall comment on some recent results concerning case B.

Let us insist upon the fact that we shall not try in this paper to formulate the most general known results, but rather to present some of the main ideas on simple examples. More general results can be found in the references which we shall give below.

2 Case A : Piecewise Monotone Observation Function.

In this section, we want to formulate the nonlinear filtering problem with small observation noise

$$\begin{cases} dX_t &= f(X_t)\,dt + g(X_t)dV_t \\ dY_t &= h(X_t)\,dt + \varepsilon dW_t \end{cases}$$

where all processes are scalar and h is piecewise monotone.

To be more specific, let us assume that $g \equiv 1$; $f, h \in C^1(\mathbb{R})$ with bounded derivatives ; h has a unique minimum at $x = 0$, such that

$$h(0) = h'(0) = 0$$

and $h'(x) < 0$ for $x < 0, h'(x) > 0$ for $x > 0$.

Remark 2.1 : *Inefficiency of the extended Kalman filter.* The extended Kalman filter for the above situation is :

$$\begin{cases} d\hat{X}_t &= f(\hat{X}_t)\,dt + \varepsilon^{-2}R_t h'(\hat{X}_t)(dY_t - h(\hat{X}_t)\,dt) \\ dR_t/dt &= 2f'(\hat{X}_t)R_t + 1 - \varepsilon^{-2}h'(\hat{X}_t)^2 R_t^2 \end{cases}$$

Note that (except possibly near $t = 0$) R_t is of the order of ε, hence $\varepsilon^{-2}R_t h'(\hat{X}_t)$ is of the order of ε^{-1}. Replacing dY_t by its expression in the first equation above yields :

$$d\hat{X}_t = [f(\hat{X}_t) + \varepsilon^{-2}R_t h'(\hat{X}_t)(h(X_t) - h(\hat{X}_t))]\,dt + \varepsilon^{-2}R_t h'(\hat{X}_t)\,dW_t$$

We note that, thanks to the leading term in the drift, the extended Kalman filter is such that $h(\hat{X}_t)$ tends to follow closely $h(X_t)$.

However this effect of the drift is counterbalanced by the noise term, which is also of the order of ε^{-1}. But the main point is that the extended Kalman filter has no tendency whatsoever to correct a wrong sign : if $\hat{X}_t$ and X_t have the same sign, the drift tends to get then closer, and if $\hat{X}_t$ and X_t have opposite signs, $\hat{X}_t$ and X_t tend to stay for away one from the other (while the drift tends to get $h(\hat{X}_t)$ and $h(X_t)$ closer). In fact, if $f(0) = 0$, then $\hat{X}_t$ never changes sign, since $h'(0) = 0$, while X_s changes sign after arbitrarily large times with positive probability.◇

Our aim is to present an efficient finite dimensional filter for the above problem in two particular cases. In order to simplify the sequel, we shall from now on assume that h is piecewise linear, i.e. :

$$h(x) = \begin{cases} h_+ x & , x \geq 0 \\ h_- x & , x \leq 0 \end{cases}$$

with $h_+ > 0, h_- < 0$. Of course, h is no longer C^1.

We want to consider cases where the variance of the conditional law of X_t, given $\{Y_s; 0 \leq s \leq t\}$ is small (at least "most" of the time). In order to see what kind of condition is needed, let us now consider the (simpler) case where $\varepsilon = 0$:

$$\begin{cases} dX_t &= f(X_t)\,dt + dV_t \\ dY_t/dt &= h(X_t) \end{cases}$$

Since $h(X_t)$ is completely observed, it suffices, in order to recover X_t, to recover its sign. We first note that we know exactly when X_t reaches 0, and when it does not change sign. Hence the problem is : given a time interval $[a,b]$ such that $X_t \neq 0$, $t \in [a,b]$, can one recover the sign of X_t, from the observation of $h(X_t)$, $t \in [a,b]$?

This is clearly impossible in the case $f \equiv 0$ and $h_- = -h_+$, since there is no way to recover the sign of a Wiener process from its absolute value. Therefore we need to introduce what we call a "detectability assumption". There are two such possible assumptions. The first one is :

$$(DA1) \quad |h_+| \neq |h_-|$$

In this case, we have that :

$$\frac{d}{dt} < h(X\cdot) >_t = h_+^2,\ t \in [a,b] \text{ if } X_t > 0,\ t \in [a,b]$$

and

$$\frac{d}{dt} < h(X.) >_t = h_-^2,\ t \in [a,b] \text{ if } X_t < 0,\ t \in [a,b]$$

In other words, under $(DA1)$, the quadratic variation of the process $\{h(X_t)\}$ tells us instantaneously the sign of X_t.

If $(DA1)$ does not hold (say $h_+ = -h_- = 1$, i.e. $h(x) = |x|$), we can still do something, provided the drift helps us. We now formulate the second detectability assumption, assuming for simplicity that $h(x) = |x|$.

$$(DA2) \quad f(x) + f(-x) \neq 0, x \in \mathbb{R}$$

Let $Z_t = h(X_t)$. If $X_t > 0$, $t \in [a,b]$, then

$$dZ_t = f(Z_t)dt + dV_t,\ t \in [a,b].$$

If $X_t < 0$, $t \in [a,b]$, then

$$dZ_t = -f(-Z_t)dt - dV_t,\ t \in [a,b]$$

i.e. we observe a Wiener process plus a drift, which differs depending on the sign of X_t, thanks to $(DA2)$. The log-likelihood ratio is given to us in this case by the Girsanov theorem : for $a \leq t \leq b$,

$$L(a,t) = \int_a^t [f(Z_s) + f(-Z_s)]dZ_s - \frac{1}{2}\int_a^t [f^2(Z_s) - f^2(-Z_s)]ds$$

Note that if $X_t > 0$, $t \in [a,b]$,

$$L(a,t) = \frac{1}{2}\int_a^t [f(Z_s) + f(-Z_s)]^2 ds + \int_a^t [f(Z_s) + f(-Z_s)]dV_s$$

and if $X_t < 0$, $t \in [a,b]$,

$$L(a,t) = -\frac{1}{2}\int_a^t [f(Z_s) + f(-Z_s)]^2 ds - \int_a^t [f(Z_s) + f(-Z_s)]dV_s$$

Hence $L(a,t)$ is likely to be positive in case $X_t > 0$, $a \le t \le b$ and to be negative in case $X_t < 0$. The quality of the test (i.e. the probability of making the right decision) depends on the value of

$$U_t = \int_a^t [f(Z_s) + f(-Z_s)]^2 ds,$$

which of course depends on $t - a$. The larger U_t is, the smaller the probability of making a wrong decision is, since from the strong law of large numbers :

$$\frac{L(a,t)}{U_t} \to \frac{1}{2} \text{ as } t \to \infty, \text{ a.s. on } \{U_\infty = +\infty\}$$

if $X_t > 0$, $t \ge a$, and the limit is $-\frac{1}{2}$ if $X_t < 0$, $t \ge a$. However, in most cases X_t changes sign after some time, and we don't want to wait too long before making a decision.

Clearly, the situation is very different under $(DA1)$ and under $(DA2)$. Under $(DA1)$, the sign of X_t is detected instantaneously, while under $(DA2)$ some time is needed for the probability of a wrong decision to be small.

Let us now describe the stragegy for a finite dimensional suboptimal filter in the case $\varepsilon > 0$. We expect that most of the time the conditional law of X_t, given $\{Y_s, 0 \le s \le t\}$ will be almost completely concentrated on one side of 0. Hence a good estimate of X_t should be given by an approximate filter for problem (2.1) with $h(x)$ replaced by $h_+ x$ (resp. $h_- x$) if $X_t > 0$ (resp. < 0). Therefore we consider the two auxiliary filtering problems :

$$(2.1_+) \quad \begin{cases} dX_t &= f(X_t)\,dt + g(X_t)\,dV_t \\ dY_t &= h_+ X_t\,dt + \varepsilon dW_t \end{cases}$$

and

$$(2.1_-) \quad \begin{cases} dX_t &= f(X_t)\,dt + g(X_t)\,dV_t \\ dY_t &= h_- X_t\,dt + \varepsilon dW_t \end{cases}$$

to which we associate, following Picard [13] the two filters :

$$(2.2_+) \quad d\hat{X}_t^+ = f(\hat{X}_t^+)\,dt + \varepsilon^{-1}(dY_t - h_+ \hat{X}_t^+\,dt)$$

$$(2.2_-)\quad d\hat{X}_t^- = f(\hat{X}_t^-)\,dt - \varepsilon^{-1}(dY_t - h_-\hat{X}_t^-\,dt)$$

The filtering procedure which we propose consists in following alternatively the filter (2.2_+) and the filter (2.2_-). Note that since these two filters are given by stiff equations, the way they are initialized at the time where we start to follow them is irrelevant. In order to choose which filter to follow, we need :

a) to isolate time–intervals on which $\{X_t\}$ is very unlikely to change sign.

b) to decide which is the sign of $\{X_t\}$ on a time interval on which we believe that this sign is fixed.

We then follow the corresponding filter until a possible zero crossing by X_t is detected.

When X_t is close to zero and/or we cannot decide its sign, we estimate it by 0.

This program has been rigorously developped under $(DA1)$ by Fleming, Ji, Pardoux [3] and Roubaud [15] in the piecewise linear case ([15] allows noise correlation and a piecewise constant diffusion coefficient) and by Fleming, Pardoux [5] in the nonlinear case with a piecewise monotone observation function. Numerical experiments are reported in Fleming, Ji, Salame, Zhang [4].

The same program has been developped under $(DA2)$ by Roubaud [15] in the piecewise linear case, and numerical experiments are reported in Milheiro, Roubaud [8]. In the next section, we shall present some of the ideas in Roubaud [15], on the above example.

3 Case A : Piecewise Monotone Observation Function under the Detectability Assumption (DA2)

We consider again the filtering problem (2.1) under the condition $(DA2)$:

$$(3.1)\quad \begin{cases} dX_t &= f(X_t)\,dt + dV_t \\ dY_t &= |X_t|\,dt + \varepsilon dW_t \end{cases}$$

with the assumption

$$(3.2) \quad \exists k \text{ s.t. } |f(x)| \le k(1+|x|),\ x \in \mathbb{R},$$

and the initial condition $X_0 = x_0$. We associate to (3.1) the two "filters" (see section 2) :

$$(3.3_+) \quad d\hat{X}_t^+ = f(\hat{X}_t^+)\,dt + \varepsilon^{-1}(dY_t - \hat{X}_t^+ dt)$$

$$(3.3_-) \quad d\hat{X}_t^- = f(\hat{X}_t^-)\,dt - \varepsilon^{-1}(dY_t + \hat{X}_t^- dt)$$

with the initial condition $\hat{X}_t^+ = x_0^+, \hat{X}_t^- = -x_0^-$.

3.1 Detection of the zero crossing by $\{X_t\}$

We first need to detect when X_t might cross zero. For that sake, we shall make use of the :

Lemma 3.1 *For any $0 < a < b$, $c > 0$, $0 \le \alpha < 1/2$ and $0 < \beta < 1 - 2\alpha$, there exist $k > 0$, $\varepsilon_0 > 0$ s.t. for any $0 < \varepsilon \le \varepsilon_0$,*

$$P(\sup_{[a,b]} ||X_t| - \hat{X}_t^+| > c\varepsilon^\alpha) \le exp(-k/\varepsilon^\beta)$$

Proof : It follows readily from (3.1), (3.2_+) and the Itô–Tanaka formula ($\{L_t, t \ge 0\}$ denotes the local time at 0 of $\{X_t\}$) :

$$d(|X_t| - \hat{X}_t^+) = \varepsilon^{-1}(|X_t| - \hat{X}_t^+)dt$$
$$+(\text{sign}(X_t)f(X_t) - f(\hat{X}_t^+))dt + \text{sign}(X_t)dV_t - dW_t + 2dL_t$$

$$\begin{aligned} |X_t| - \hat{X}_t^+ &= e^{-t/\varepsilon}(|X_0| - \hat{X}_0^+) \\ &+ \int_0^t e^{-(t-s)/\varepsilon}(\text{sign}(X_s)f(X_s) - f(\hat{X}_s^+))\,ds \\ &+ \int_0^+ e^{-(t-s)/\varepsilon}\text{sign}(X_s)\,dV_s - \int_0^t e^{-(t-s)/\varepsilon}dW_s \\ &+2\int_0^t e^{-(t-s)/\varepsilon}dL_s \end{aligned}$$

It suffices to establish the result for each of the terms in the above right side. The four first terms are treated as in [5, Lemma 3.1] with the help of Lemma 3.2 below. The last term is analysed as in [15, Proposition I.3.1].◇

Lemma 3.2 *For any $0 < a < b$, there exists $c > 0$ s.t.*

$$E\left\{\sup_{a\leq t\leq b} \exp[cX_t^2]\right\} < \infty$$

$$\sup_{\varepsilon>0} E\left\{\sup_{a\leq t\leq b} \exp[c(\hat{X}_t^+)^2]\right\} < \infty$$

Proof : The first result is well-known (see e.g. [15, Lemma I.2.7]). The second one can be established as follows :

$$d\hat{X}_t^+ = f(\hat{X}_t^+)\,dt + \varepsilon^{-1}(|X_t| - \hat{X}_t^+)\,dt + dW_t$$

For $x \in I\!R$ and $\{Z_t\}$ a bounded variation process, let $U_t(x,z)$ denote the solution of :

$$U_t(x,z) = x + \int_0^t f(U_s(x,z))\,ds + W_t + Z_t$$

Let $U_t^M = U_t(x_0, Z^M)$, where $\{Z_t^M\}$ is the smallest increasing process s.t.

$$U_t(x_0, Z^M) \geq |X_t|,\ t \geq 0,$$

and $U_t^m = U_t(x_0, Z^m)$, where $\{Z_t^m\}$ is the largest decreasing process s.t.

$$U_t(x_0, Z^m) \leq |X_t|,\ t \geq 0.$$

It follows from a comparison theorems for one dimensional SDEs that

$$U_t(x_0, Z^m) \leq \hat{X}_t^+ \leq U_t(x_0, Z^M)$$

and

$$E(\sup_{t\in[a,b]} \exp[cU_t^2(x_0, Z^m)] + \sup_{t\in[a,b]} \exp[cU_t^2(x_0, Z^M)]) < \infty$$

follows from the same result for $\{|X_t|\}$. Note that another proof of the second part of this Lemma, which carries over to higher dimension, is given in [15, Lemma I.2.8].◇

With the help of Lemma 3.1, we can easily build a procedure which detects the possible zero crossings by $\{X_t\}$.

We choose $\bar{c} > 0$ and $0 \le \alpha < 1/2$. Whenever $\hat{X}_t^+ \le \bar{c}\varepsilon^\alpha$, we conclude that X_t might be zero, and we choose 0 as the estimate of X_t. Whenever $\hat{X}_t^+ > \bar{c}\varepsilon^\alpha$, we decide that $X_t \neq 0$ and we try to estimate its sign. For any $0 \le a < b$, we define

$$C(a,b) = \{\hat{X}_t^+ > \bar{c}\varepsilon^\alpha, a \le t \le b\}$$

The next and last step consists in a test for deciding, conditionned upon the observation to belong to $C(a,b)$, upon the sign of $X_t, t \in [a,b]$.

There are two possible tests for this problem. The first one is an extrapolation of the test used in the case $\varepsilon = 0$, and the second one is a likelihood ratio test based on the outputs of the two filters (3.3_+) and (3.3_-).

Before presenting those two tests, let us formulate a stronger version of $(DA2)$, which will be supposed to hold throughout the rest of this section :

$$(DA2s) \quad \exists \; c, d > 0 \text{ s.t.} \inf_{|x-y|\le c} [f(x) + f(y)] \ge d$$

3.2 Deciding the Sign of X_t : a Test based on the Increments of $\{Y_t\}$.

Define

$$F(x) = \int_0^x [f(y) + f(-y)]dy$$

$L(a,b)$, which was defined in section 2, can be rewritten as :

$$\begin{aligned} L(a,b) &= F(Z_b) - F(Z_a) + \frac{1}{2}\int_a^b [f'(-Z_t) - f'(Z_t)]dt \\ &+ \frac{1}{2}\int_a^b [f^2(-Z_t) - f^2(Z_t)]dt \end{aligned}$$

Of course, in the case $\varepsilon > 0$, $Z_t = h(X_t)$ is not observed, and $L(a,b)$ is no longer a statistics. We note that :

$$\begin{aligned}\varepsilon^{-1}(Y_{t+\varepsilon}-Y_t) &= \varepsilon^{-1}\int_t^{t+\varepsilon} Z_s ds + W_{t+\varepsilon} - W_t \\ &\simeq Z_t + W_{t+\varepsilon} - W_t\end{aligned}$$

Let $m = [\varepsilon^{-1}(b-a)]$ ($[\cdot]$ denotes the integer part of its argument), $t_k = a + k\varepsilon$, $k = 0, 1, \ldots, m$,

$$\bar{Z}_k = \varepsilon^{-1}(Y_{t_{k+1}} - Y_{t_k}),\ k = 0, 1, \ldots, m-1$$

We can then define the statistics :

$$\begin{aligned}L^\varepsilon(a,b) &= F(\bar{Z}_{m-1}) - F(\bar{Z}_0) \\ &+ \varepsilon/2 \sum_{k=0}^{m-1} [f'(-\bar{Z}_k) - f'(\bar{Z}_k) + f^2(-\bar{Z}_k) - f^2(\bar{Z}_k)]\end{aligned}$$

Assuming in addition to the above hypotheses that f' is Lipschitz, it is not hard to show that for small $\varepsilon > 0$ $L^\varepsilon(a,b)$ is close to $L(a,b)$. Hence, if $L^\varepsilon(a,b) > 0$ (resp. < 0), and provided the observation belongs to $C(a,b)$ and $\int_a^b [f(Z_t) + f(-Z_t)]^2 dt$ is large enough, there is a high probability that $X_t > 0$ (resp. < 0), $t \in [a,b]$.

Again, the details can be found in [15]. We note that the extension of this method to higher dimension requises that f be a gradient.

3.3 A Likelihood Ratio Test based on the Outputs of the two Approximate Linear Filters.

We consider again the filtering problem (3.1), to which we associate the two approximate linear filters (3.3_+) and (3.3_-). Note that, if $\mathcal{Y}_t = \sigma\{Y_s; 0 \le s \le t\}$,

$$dY_t = E(|X_t|/\mathcal{Y}_t)dt + \varepsilon d\nu_t$$

where $\{\nu_t\}$, the innovation process, is a standard Wiener process (see e.g.[9]). We expect that if $X_t > 0$ (resp. < 0), $t \in [a,b]$, then $\hat{X}_t^+$ (resp. $\hat{X}_t^-$) is very close to $E(|X_t|/\mathcal{Y}_t)$, at least after some time.

Therefore we introduce the following quantity, which we interpret as being an approximate log-likelihood ratio. Let $a < e < b$. Define

$$\hat{L}^\varepsilon = \varepsilon^{-2}\int_e^b (\hat{X}_t^+ + \hat{X}_t^-)dY_t - \varepsilon^{-2}/2\int_e^b (|\hat{X}_t^+|^2 - |\hat{X}_t^-|^2)dt$$

We shall now show that, provided $e-a$ is not too small and $b-e$ is large enough, the conditional probability

$$P(\hat{L}^\varepsilon > 0/A_+(a,b)),$$

where $A_+(a,b) = \{X_t > 0, a < t < b\}$, is very close to one. A similar result holds for $P(\hat{L}^\varepsilon < 0/A_-(a,b))$. let

$$M_t^+ = \exp[\varepsilon^{-1}\int_a^{a\vee t}(X_s - |X_s|)dW_s - \varepsilon^{-2}/2\int_a^{a\vee t}(X_s - |X_s|)^2 ds]$$

and P^+ be a new probability measure given by :

$$\frac{dP^+}{dP} = M_b^+$$

From Girsanov's theorem,

$$dY_t = X_t dt + \varepsilon dW_t^+,\ 0 \le t \le b$$

where $\{W_t^+, 0 \le t \le b\}$ is a standard Wiener process under P^+. Hence, again from well-known results from nonlinear filtering, if

$$\tilde{X}_t^+ = E^+(X_t/\mathcal{Y}_t),\ 0 \le t \le b$$

then

$$dY_t = \tilde{X}_t^+ dt + \varepsilon d\nu_t^+, 0 \le t \le b$$

We have

$$\begin{aligned}\hat{L}^\varepsilon &= \varepsilon^{-1}\int_e^b (\hat{X}_t^+ + \hat{X}_t^-)d\nu_t^+ + \varepsilon^{-2}/2\int_e^b (\hat{X}_t^+ + \hat{X}_t^-)^2 dt \\ &+ \varepsilon^{-2}\int_e^b (\hat{X}_t^+ + \hat{X}_t^-)(\tilde{X}_t^+ - \hat{X}_t^+)dt\end{aligned}$$

We shall show that on $A_+(a,b)$ the third term above is negligible, compared to the second term. Hence the sign of L^ε is given by that of

$$\begin{aligned}\hat{L}^{\varepsilon,+} &= \varepsilon^{-1}\int_e^b (\hat{X}_t^+ + \hat{X}_t^-)d\nu_t^+ + \varepsilon^{-2}/2\int_e^b (\hat{X}_t^+ + \hat{X}_t^-)^2 dt \\ &= N_b + 1/2 < N >_b\end{aligned}$$

where $\{N_t\}$ is a P^+ martingale. Hence $P^+(\hat{L}^{\varepsilon,+} > 0/ < N >_b > r)$ is close to one when r is large. We now establish :

Lemma 3.3 *For any $0 < \beta < 1$, $r < (b-e)d^2$, there exists k, $\varepsilon_0 > 0$ s.t. for any $\varepsilon \in [0, \varepsilon_0]$,*

$$P(\varepsilon^{-2}\int_e^b (\hat{X}_t^+ + \hat{X}_t^-)^2 dt < r) \leq \exp(-k/\varepsilon^\beta)$$

Proof : From lemma 3.1 (and its analogue with $\hat{X}_t^+$ replaced by $-\hat{X}_t^-$), for any $\beta < 1$, there exists k, ε_0 s.t. :

$$P(\{\sup_{[a,b]} ||X_t| - \hat{X}_t^+| > \frac{c}{2}\} \cup \{\sup_{[a,b]} ||X_t| + \hat{X}_t^-| > \frac{c}{2}\}) \leq exp(-k/\varepsilon^\beta)$$

However,

$$\frac{d}{dt}(\hat{X}_t^+ + \hat{X}_t^-) = -\varepsilon^{-1}(\hat{X}_t^+ + \hat{X}_t^-) + f(\hat{X}_t^+) + f(\hat{X}_t^-)$$

hence

$$\hat{X}_t + \hat{X}_t^- = e^{-(t-a)/\varepsilon}(\hat{X}_a^+ + \hat{X}_a^-) + \int_a^t e^{-(t-s)/\varepsilon}[f(\hat{X}_s^+) + f(\hat{X}_s^-)]ds$$

The first term in the above right side is very small for $t \geq e$. The second term is bounded below by :

$$\varepsilon(1 - e^{-(e-a)/\varepsilon}) \inf_{a\leq t\leq b}[f(\hat{X}_t^+) + f(\hat{X}_t^-)]$$

Provided $|\hat{X}^+ + \hat{X}_t^-| \leq c$, we deduce from (DA 2s) that :

$$f(\hat{X}_t^+) + f(\hat{X}_t^-) \geq d$$

The result now follows. ◊

The fact that the term

$$\varepsilon^{-2}\int_{e}^{b}(\hat{X}_t^+ + \hat{X}_t^-)(\tilde{X}_t^+ - \hat{X}_t^+)dt$$

is negligible on $A^+(a,b)$ follows from Lemma 3.3, the obvious remark that P^+ and P coïncide on $A^+(a,b)$ (since $M_b^+ = 1$ on this set) and Theorem 3 from Picard [13] which states that for any $p \geq 1$,

$$(E^+[|\hat{X}_t^+ - \tilde{X}_t^+|^p])^{1/p} = 0(\varepsilon^{3/2})$$

We can easily conclude from the above :

Proposition 3.1 *There exists a continuous decreasing function ρ : $\mathbb{R}_+ \to \mathbb{R}_+$, with $\lim_{x\to+\infty}\rho(x) = 0$, and for any $p \geq 1$, there exist k, $\varepsilon_0 > 0$ s.t. :*

$$P(\{L^\varepsilon < 0\} \cap A_+(a,b) \cap C(a,b)) \leq k\varepsilon^p + \rho(b-a)$$

for any $\varepsilon \in (0, \varepsilon_0]$. ◊

We note that the difference with the results under (DA 1) is the appearance of the term $\rho(b-a)$: for a fixed interval [a,b], under (DA 2) the probability of making a wrong decision does not tend to zero as $\varepsilon \downarrow 0$. This is consistent with the results is the $\varepsilon = 0$ case.

4 Remarks on the problems with $\dim X > \dim Y$

Suppose that $\dim X = 2$ and $\dim Y = 1$, and that

$$dY_t = h(X_t^1)\,dt + \varepsilon dW_t$$

Assume first that h is monotone. Then one can show (see Yaesh, Bobrovsky, Schuss [16], Picard [14]) that there exist efficient linearized filters, provided that the variance of the conditional law of X_t, given $\{Y_s, 0 \leq s \leq t\}$ is small. But now this need not be the case

in general. It is the case for the following model, which has been rigorously analysed by Milheiro [7].

$$\begin{cases} dX_t^1 &= f_1(X_t^1, X_t^2)dt \\ dX_t^2 &= f_2(X_t^1, X_t^2)dt + g(X_t^1, X_t^2)dV_t \\ dY_t &= h(X_t^1)dt + \varepsilon dW_t \end{cases}$$

where f is C^2, for each $x_1, x_2 \to f_1(x_1, x_2)$ is one to one and its inverse is Lipschitz, and some further regularity assumptions are satisfied. Milheiro [7] gives a two dimensional filter with output $\bar{X}_t$ which is such that, for $t \geq t_0 > 0$,

$$E(|X_t^1 - \bar{X}_t^1|^2) = 0(\varepsilon^{3/2}), \quad E(|X_t^2 - \bar{X}_t^2|^2) = 0(\varepsilon^{1/2})$$

It is also possible to derive approximate finite dimensional filters, even when the covariance of the conditional law is not small with ε, provided the problem has a special structure. Let us describe a problem which has been successfuly treated in Roubaud [15]. Again, $\dim X = 2$ and $\dim Y = 1$. We assume that :

$$\begin{cases} dX_t &= (f(X_t^1) + bX_t^2)\, dt + G dV_t \\ dY_t &= h(X_t^1)\, dt + \varepsilon dW_t \end{cases}$$

where $f \; : \mathbb{R} \to \mathbb{R}^2$ and $h \; : \mathbb{R} \to \mathbb{R}$ are piecewise–linear (with say two pieces), h being non monotone, $b \in \mathbb{R}^2$, G is a 2×2 matrix, and $\{V_t\}$ is two-dimensional standard Wiener process. As in the preceding section, we associate to this problem two (linear) filters, and test procedures to decide which filter to follow. The conditional variance in the x^1 direction is small, but it is in general of order one in the x^2 direction. The fact that the system is linear in X_t^2 is crucial here. Note that one major difference with the situation of the preceding section is that the filter (or at least its second component) does not have a short memory.

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LIKELIHOOD BASED STATISTICS FOR PARTIALLY OBSERVED DIFFUSION PROCESSES*

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Abstract

The purpose of this paper is to study some statistical problems : parameter estimation, binary detection, change detection (disorder problem), etc. for partially observed diffusion processes, using the likelihood approach.

It is shown that the stochastic PDE related to the state estimation problem, provides also a way to compute the likelihood function/ratio.

A recent result on consistency of the MLE, in the small noise asymptotics, is also presented.

1 Introduction

Consider the following partially observed stochastic differential system, defined on some probability space

$$\begin{aligned} dX_t &= b(X_t)\,dt + \sigma(X_t)\,dW_t \\ dY_t &= h(X_t)\,dt + dV_t \end{aligned} \tag{1}$$

where the non observed process $\{X_t\,, t \geq 0\}$ and the observation $\{Y_t\,, t \geq 0\}$ takes values in $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively. $\{W_t\,, t \geq 0\}$ and $\{V_t\,, t \geq 0\}$ are independent Wiener processes of appropriate dimension, with covariance matrix I, and the random variable X_0 is independent of the Wiener processes, with probability distribution $p_0(x)\,dx$. The available information at time t is contained in the σ-algebra $\mathcal{Y}_t = \sigma(Y_s\,, 0 \leq s \leq t)$.

The first problem one is faced with, is state estimation : to estimate recursively the state X_t given observations $\mathcal{Y}_t$ up to time t. The solution to this first problem is given by the Zakai equation, a stochastic partial differential equation which computes recursively the conditional density of X_t given $\mathcal{Y}_t$. This assumes that the partially observed dynamical system (1) is completely identified, which usually is not the case.

Therefore, a second problem is to assume that the model (1) is parametrized by some unknown parameter θ in $\Theta \subset \mathbf{R}^p$, and to estimate θ on the basis of observations $\mathcal{Y}_t$. Several statistical problems are introduced in Section 2 for the parametrized model (1). Off-line statistical procedures based on likelihood are presented in Section 3. It is shown in Section 4 that the Zakai equation provides also a way to compute these likelihood statistics.

Another issue is to prove that the statistical algorithms based on the likelihood approach, actually provide good estimates, in some asymptotic sense. A recent result in this direction has been obtained for the consistency of the MLE, in the small noise asymptotics, see James-LeGland [4].

2 Statistical problems

Let $\Theta \subset \mathbf{R}^p$ denote the parameter space. Assume that observations $\{Y_t\,, 0 \leq t \leq T\}$ are available from the following model

$$\begin{aligned} dX_t &= b_\theta(X_t)\,dt + \sigma(X_t)\,dW_t^\theta \\ dY_t &= h_\theta(X_t)\,dt + dV_t^\theta \end{aligned}$$

The statistical problems to be considered in this paper are

(a) parameter estimation : estimate $\theta \in \Theta$.

(b) binary detection : decide between the two simple hypotheses

$$\mathrm{H}_0 : \quad \theta = \theta_0\,,$$

$$\mathrm{H}_1 : \quad \theta = \theta_1\,.$$

Another related problem is the sequential binary detection problem.

(c) change detection (disorder problem) : decide be-

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tween the two composite hypotheses

$$\mathrm{H}_0 : \quad \theta = \theta_0 ,$$

H_1 : there exists $0 \leq r \leq T$, such that

$$\begin{cases} \theta = \theta_0 & \text{on} \quad 0 \leq t < r , \\ \theta = \theta_1 & \text{on} \quad r \leq t \leq T . \end{cases}$$

In case H_1 has been decided, another problem of interest is to estimate the change time r.

A variant of this problem, is when only θ_0 is known : the alternate hypothesis H_1 is composite with respect to both r and θ_1. In case H_1 has been decided, both r and θ_1 are to be estimated.

(d) Bayesian change detection (jump Markov parameter) : recursively estimate θ_t given $\mathcal{Y}_t$, assuming that $\{\theta_t , t \geq 0\}$ is a finite state Markov process, independent of the Wiener processes, with jump intensity matrix Q. This problem is closely related to state estimation, see Loparo–Roth–Eckert [7].

For each of the problems listed above, the first step is to provide an expression, in terms of conditional expectation, for the likelihood function (LF), the likelihood ratio (LR), or the generalized likelihood ratio (GLR), depending on the problem.

3 Likelihood based off–line statistics

Statistical model On the canonical space $\Omega = C([0,T] ; \mathbf{R}^{m+d})$ are given

- a pair of stochastic processes $\{X_t , 0 \leq t \leq T\}$ and $\{Y_t , 0 \leq t \leq T\}$ taking values in $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively,
- a family $\mathcal{M} = \{P_\theta , \theta \in \Theta\}$ of probability measures,

such that under P_θ

$$\begin{aligned} dX_t &= b_\theta(X_t)\, dt + \sigma(X_t)\, dW_t^\theta \\ dY_t &= h_\theta(X_t)\, dt + dV_t^\theta \end{aligned} \tag{2}$$

where $\{W_t^\theta , 0 \leq t \leq T\}$ and $\{V_t^\theta , 0 \leq t \leq T\}$ are independent Wiener processes of appropriate dimension, with covariance matrix I, and the random variable X_0 is independent of the Wiener processes, with probability distribution $p_0^\theta(x)\, dx$. Throughout the paper, the coefficients are assumed to be continuous and bounded functions on $\mathbf{R}^m$.

The main assumption is that all the available information is contained in $\mathcal{Y}_T = \sigma(Y_t , 0 \leq t \leq T)$.

Introduce

$$Z_t^s[\theta] \triangleq \exp\left\{ \int_s^t h_\theta^*(X_\tau)\, dY_\tau - \tfrac{1}{2} \int_s^t |h_\theta(X_\tau)|^2\, d\tau \right\}$$

and $Z_t[\theta] \triangleq Z_t^0[\theta]$. Provided the probability measures on $\mathbf{R}^m$ with densities $\{p_0^\theta , \theta \in \Theta\}$ are mutually absolutely continuous, the statistical model defined above is dominated by some probability measure $P^\dagger$. Indeed, it is proved in [2] that

Proposition 3.1 *The probability measures in $\mathcal{M}$ are mutually absolutely continuous. In addition*

$$\left.\frac{dP_\theta}{dP^\dagger}\right|_{\mathcal{Y}_T} = \mathbf{E}_\theta^\dagger(Z_T[\theta] \mid \mathcal{Y}_T) ,$$

where $P_\theta^\dagger$ is the reference probability *measure.*

Parameter estimation

The likelihood function for the estimation of θ, based on observations in $\mathcal{Y}_T$, is given by

$$L[\theta] \triangleq \left.\frac{dP_\theta}{dP^\dagger}\right|_{\mathcal{Y}_T} = \mathbf{E}_\theta^\dagger(Z_T[\theta] \mid \mathcal{Y}_T) , \tag{3}$$

and the maximum likelihood estimate (MLE) is defined as

$$\widehat{\theta} \in \operatorname*{argmax}_{\theta\in\Theta} L[\theta] .$$

To find $\widehat{\theta}$, one can use an iterative optimization algorithm for the maximization of the likelihood function $L[\theta]$. An alternative approach is to use the EM algorithm, as proposed by Dembo–Zeitouni [3]. This algorithm is based on the following immediate consequence of the Jensen inequality, where $\ell[\theta]$ denotes the log-likelihood function

$$\begin{aligned} \ell[\theta] - \ell[\theta'] &= \log \mathbf{E}_{\theta'}^\dagger\Big(\frac{Z_T[\theta]}{Z_T[\theta']} \mid \mathcal{Y}_T\Big) \\ &\geq \mathbf{E}_{\theta'}^\dagger\Big(\log \frac{Z_T[\theta]}{Z_T[\theta']} \mid \mathcal{Y}_T\Big) \triangleq Q[\theta,\theta'] . \end{aligned}$$

The idea of the EM algorithm is to replace the direct maximization of $L[\theta]$, by the iterative maximization of the auxiliary function, i.e.

$$\widehat{\theta}^{n+1} \in \operatorname*{argmax}_{\theta\in\Theta} Q[\theta,\widehat{\theta}^n] .$$

Under mild hypotheses, the sequence $\{\widehat{\theta}^n , n \geq 0\}$ converges to a stationary point of the original likelihood function $L[\theta]$. See Campillo–LeGland [2] for a comparison between the two approaches.

Binary detection

The likelihood ratio for deciding between hypotheses H_0 and H_1, based on observations in $\mathcal{Y}_T$, is given by

$$R \triangleq \left.\frac{dP_1}{dP_0}\right|_{\mathcal{Y}_T} = \frac{L[\theta_1]}{L[\theta_0]} , \tag{4}$$

where $P_i = P_{\theta_i}$ for $i = 0, 1$. The likelihood ratio test is defined by the following reject region for the null hypothesis H_0

$$R \geq c \ ,$$

where $c > 1$ is the threshold.

Sequential binary detection

In this problem, the horizon is not fixed. Let R_t denote the likelihood ratio for deciding between hypotheses H_0 and H_1, based on observations in $\mathcal{Y}_t$. An *admissible decision policy* for the sequential binary detection problem, is defined by a stopping time τ and a $\mathcal{Y}_\tau$–measurable $\{0,1\}$–valued decision random variable δ : if $\delta = 0$ (resp. $\delta = 1$) the null hypothesis H_0 is accepted (resp. rejected). In other words, δ defines a reject region for the null hypothesis H_0. A *threshold decision policy* is defined by a stopping time of the form

$$\tau \triangleq \inf\{t \geq 0 \ : \ R_t \notin (A, B)\}$$

and a reject region for the null hypothesis H_0 of the form

$$\delta = \begin{cases} 1 \ , & \text{if } R_\tau \geq B \ , \\ 0 \ , & \text{if } R_\tau \leq A \end{cases}$$

where $0 < A < 1 < B < \infty$ are the constant thresholds. This problem has been studied by Baras–LaVigna [1], following Liptser–Shiryayev [6].

Change detection (disorder)

The statistical model for this problem can be described through the introduction of time dependent coefficients : for $0 \leq r \leq T$, let P_r denote the probability measure on the canonical space Ω, under which

$$\begin{aligned} dX_t &= b_r(t, X_t)\, dt + \sigma(X_t)\, dW_t^r \\ dY_t &= h_r(t, X_t)\, dt + dV_t^r \end{aligned} \tag{5}$$

where $\{W_t^r \ , 0 \leq t \leq T\}$ and $\{V_t^r \ , 0 \leq t \leq T\}$ are independent Wiener processes of appropriate dimension, with covariance matrix I, and

$$b_r(t,x) \triangleq b_0(x) + [b_1(x) - b_0(x)]\, 1_{\{r \leq t \leq T\}} \ ,$$

$$h_r(t,x) \triangleq h_0(x) + [h_1(x) - h_0(x)]\, 1_{\{r \leq t \leq T\}} \ ,$$

where $b_i(x) = b_{\theta_i}(x)$ and $h_i(x) = h_{\theta_i}(x)$ for $i = 0, 1$

Introducing for $0 \leq r \leq T$

$$Z_t^s[r] \triangleq \exp\left\{ \int_s^t h_r^*(\tau, X_\tau)\, dY_\tau - \tfrac{1}{2} \int_s^t |h_r(\tau, X_\tau)|^2\, d\tau \right\}$$

and $Z_t[r] \triangleq Z_t^0[r]$, it holds that the probability measures $\{P_r^\dagger \ , 0 \leq r \leq T\}$ are mutually absolutely continuous. Moreover

$$\left.\frac{dP_r}{dP^\dagger}\right|_{\mathcal{Y}_T} = \mathbf{E}_r^\dagger(Z_T[r] \mid \mathcal{Y}_T) \ .$$

Note that, with this definition, the probability associated with the null hypothesis H_0 is P_T.

The generalized likelihood ratio for deciding between hypotheses H_0 and H_1, based on observations in $\mathcal{Y}_T$, is given by

$$R \triangleq \max_{0 \leq r \leq T} \left.\frac{dP_r}{dP_T}\right|_{\mathcal{Y}_T} = \max_{0 \leq r \leq T} \frac{L[r]}{L[T]} \ , \tag{6}$$

where

$$L[r] \triangleq \left.\frac{dP_r}{dP^\dagger}\right|_{\mathcal{Y}_T} = \mathbf{E}_r^\dagger(Z_T[r] \mid \mathcal{Y}_T) \ ,$$

is the likelihood function for the estimation of the change time r, based on observations in $\mathcal{Y}_T$. The generalized likelihood ratio test is defined by the following reject region for the null hypothesis H_0

$$R \geq c \ ,$$

where $c > 1$ is the threshold.

Moreover, in case H_1 has been decided, the maximum likelihood estimate of the change time r, based on observations in $\mathcal{Y}_T$, is given by

$$\hat{r} \in \operatorname*{argmax}_{0 \leq r \leq T} L[r] \ .$$

Introducing the σ–algebras $\mathcal{F}_t = \sigma(X_s \ , 0 \leq s \leq t)$ and $\mathcal{Y}_t^s = \sigma(Y_\tau - Y_s \ , s \leq \tau \leq t)$, the following decomposition holds for the likelihood function $L[r]$

$$\begin{aligned} L[r] &= \mathbf{E}_r^\dagger(Z_T[r] \mid \mathcal{Y}_T) = \mathbf{E}_r^\dagger(Z_r[0] \cdot Z_T^r[1] \mid \mathcal{Y}_T) \\ &= \mathbf{E}_r^\dagger\left(\mathbf{E}_r^\dagger(Z_r[0] \cdot Z_T^r[1] \mid \mathcal{F}_r \vee \mathcal{Y}_r \vee \mathcal{Y}_T^r) \mid \mathcal{Y}_T\right) \\ &= \mathbf{E}_r^\dagger\left(Z_r[0] \cdot \mathbf{E}_r^\dagger(Z_T^r[1] \mid \mathcal{F}_r \vee \mathcal{Y}_T^r) \mid \mathcal{Y}_T\right) \\ &= \mathbf{E}_r^\dagger\left(Z_r[0] \cdot \mathbf{E}_1^\dagger(Z_T^r[1] \mid \mathcal{F}_r \vee \mathcal{Y}_T^r) \mid \mathcal{Y}_T\right) \ , \end{aligned}$$

where $Z_t^s[i] = Z_t^s[\theta_i]$ for $i = 0, 1$. Defining

$$v_t^1(x) \triangleq \mathbf{E}_1^\dagger(Z_T^t[1] \mid \mathcal{Y}_T^t \vee \{X_t = x\}) \ ,$$

it holds

$$\begin{aligned} L[r] &= \mathbf{E}_r^\dagger(Z_r[0] \cdot v_r^1(X_r) \mid \mathcal{Y}_T) \\ &= \mathbf{E}_0^\dagger(Z_r[0] \cdot v_r^1(X_r) \mid \mathcal{Y}_T) \ . \end{aligned} \tag{7}$$

The purpose of the next section is to provide some computational procedure, that would allow to numerically compute the likelihood based statistics introduced so far.

4 Computational likelihood statistics

In this section, the link between the likelihood based statistical problems introduced above, and the

state estimation problem, will be investigated. At this point, it is necessary to introduce some notations and definitions related to nonlinear filtering and smoothing.

For the sake of simplicity, any reference to the parameter θ will be dropped for the time being.

□ *Filtering:* Let p_t denote the unnormalized conditional density of the random variable X_t given $\mathcal{Y}_t$, defined by

$$(p_t, \phi) \triangleq \mathbf{E}^{\dagger}(\phi(X_t)Z_t \mid \mathcal{Y}_t) \tag{8}$$

for any test-function ϕ. The unnormalized conditional density $\{p_t\,,\, t \geq 0\}$ satisfies the Zakai equation [8]

$$dp_t = L^* p_t\, dt + h^* p_t\, dY_t\ , \tag{9}$$

where L is the following partial differential operator, associated with the stochastic differential system (1)

$$L \triangleq \tfrac{1}{2} \sum_{i,j=1}^{m} a^{i,j}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b^i(\cdot) \frac{\partial}{\partial x_i}\ .$$

□ *Smoothing (fixed-interval):* Let $T > 0$ denote the fixed end-time, and q_t denote the unnormalized conditional density of the random variable X_t given $\mathcal{Y}_T$, defined by

$$(q_t, \phi) \triangleq \mathbf{E}^{\dagger}(\phi(X_t)Z_T \mid \mathcal{Y}_T)\ .$$

Introducing the backward Zakai equation

$$dv_t + Lv_t\, dt + h^* v_t\, dY_t = 0\ , \qquad v_T \equiv 1\ , \tag{10}$$

it is proved in Pardoux [8,9] that (p_t, v_t) is independent of t and $q_t = p_t \cdot v_t$. In addition

$$v_t(x) = \mathbf{E}^{\dagger}(Z_T^t \mid \mathcal{Y}_T^t \vee \{X_t = x\})\ .$$

Existence, uniqueness and regularity results for stochastic PDE can be found in Pardoux [8].

Let now L_θ and $L_r(t)$ denote the partial differential operators

$$L_\theta \triangleq \tfrac{1}{2} \sum_{i,j=1}^{m} a^{i,j}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_\theta^i(\cdot) \frac{\partial}{\partial x_i}\ ,$$

$$L_r(t) \triangleq \tfrac{1}{2} \sum_{i,j=1}^{m} a^{i,j}(\cdot) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b_r^i(t,\cdot) \frac{\partial}{\partial x_i}\ ,$$

associated with the stochastic differential equation (2) and (5) respectively.

Parameter estimation

The following expression holds for the likelihood function (3)

$$L[\theta] = (p_T^\theta, 1)\ ,$$

where the unnormalized conditional density $\{p_t^\theta\,,\, t \geq 0\}$ solves the Zakai equation

$$dp_t^\theta = L_\theta^* p_t^\theta\, dt + h_\theta^* p_t^\theta\, dY_t\ .$$

Binary detection

A similar expression holds for the likelihood ratio (4)

$$R = \frac{(p_T^1, 1)}{(p_T^0, 1)}\ .$$

Here the unnormalized conditional density $\{p_t^i\,,\, t \geq 0\}$ solves the Zakai equation

$$dp_t^i = L_i^* p_t^i\, dt + h_i^* p_t^i\, dY_t\ , \tag{11}$$

where $L_i = L_{\theta_i}$ and $h_i = h_{\theta_i}$ for $i = 0, 1$.

Change detection (disorder)

Let $\{p_t^r\,,\, t \geq 0\}$ and $\{v_t^r\,,\, t \geq 0\}$ denote the solution of

$$dp_t^r = L_r^*(t) p_t^r\, dt + h_r^*(t) p_t^r\, dY_t\ ,$$

and

$$dv_t^r + L_r(t) v_t^r\, dt + h_r^*(t) v_t^r\, dY_t = 0\ , \qquad v_T^r \equiv 1\ ,$$

respectively, where $h_r(t)$ is shorthand for $h_r(t,\cdot)$. The generalized likelihhod ratio (6) is given by

$$R = \max_{0 \leq r \leq T} \frac{(p_T^r, 1)}{(p_T^T, 1)}\ . \tag{12}$$

However, a much more efficient expression can be obtained. Indeed, for all $0 \leq t \leq T$

$$L[r] = (p_T^r, 1) = (p_t^r, v_t^r)\ ,$$

and in particular for $t = r$

$$L[r] = (p_r^r, v_r^r) = (p_r^0, v_r^1)\ , \tag{13}$$

where

$$dp_t^0 = L_0^* p_t^0\, dt + h_0^* p_t^0\, dY_t\ , \tag{14}$$

and

$$dv_t^1 + L_1 v_t^1\, dt + h_1^* v_t^1\, dY_t = 0\ , \qquad v_T^1 \equiv 1\ . \tag{15}$$

Therefore, it is enough to solve two stochastic PDE, the forward equation (14) with parameter θ_0, and the backward equation (15) with parameter θ_1. This gives the following expression for the generalized likelihood ratio

$$R = \max_{0 \leq r \leq T} \frac{(p_r^0, v_r^1)}{(p_T^0, 1)}\ .$$

which is much more efficient than the original expression (12), which would require to solve an infinite number of stochastic PDE (see Figure 1).

Remark 4.1 The expression (13) for the likelihood ratio could also be obtained from the previous expression (7) obtained by decomposition.

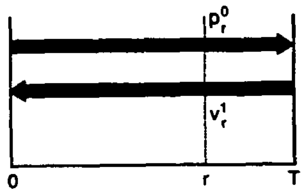

Figure 1: Stochastic PDE for the disorder problem.

It can also be proved that the likelihood function $r \mapsto L[r]$ is smooth, provided the change can only occur in the drift coefficient, i.e. $h_0 = h_1$. Actually, using the two-sided stochastic calculus developped in Pardoux [10]

$$\begin{aligned} d(p^0_t, v^1_t) &= (L^*_0 p^0_t, v^1_t)\,dt + (h^*_0 p^t_0, v^1_t)\,dY_t \\ &\quad - (p^0_t, L_1 v^1_t)\,dt - (p^0_t, h^*_0 v^1_t)\,dY_t \\ &= (p^0_t, [L_0 - L_1]\, v^1_t)\,dt\ . \end{aligned}$$

Bayesian change detection

The unnormalized conditional distribution of the compound state (X_t, θ_t) given observations in $\mathcal{Y}_t$, is defined by

$$(p^i_t, \phi) = \mathbf{E}^\dagger(\phi(X_t)\, 1_{\{\theta_t = i\}}\, Z_t[\theta] \mid \mathcal{Y}_t)\ ,$$

for $i = 1, 2, \ldots, N$, where in this section the process $\{Z_t[\theta]\,,\, 0 \le t \le T\}$ is defined by

$$Z^s_t[\theta] \triangleq \exp\left\{ \int_s^t h^*_{\theta_r}(X_r)\, dY_r - \tfrac{1}{2} \int_s^t |h_{\theta_r}(X_r)|^2\, dr \right\}$$

and $Z_t[\theta] \triangleq Z^0_t[\theta]$. In addition $\{p^i_t\,,\, 0 \le t \le T\}$ satisfies the following system of coupled Zakai equations

$$dp^i_t = L^*_i p^i_t\, dt + h^*_i p^i_t\, dY_t + \sum_{j=1}^N q_{j,i}\, p^j_t\, dt\ , \qquad (16)$$

where $Q = \{q_{i,j}\}$ is the jump intensity matrix for the Markov process $\{\theta_t\,,\, 0 \le t \le T\}$. Note that in system (16) the coupling occurs only through zero-order state-independent coefficients.

The unnormalized marginal conditional distribution

$$(p^i_t, 1) = c_t \cdot P(\theta_t = i \mid \mathcal{Y}_t)\ ,$$

can be used to compute the maximum a posteriori (MAP) estimate

$$\widehat{\theta}^{MAP}_t \in \underset{1 \le i \le N}{\operatorname{argmax}}\, (p^i_t, 1)\ .$$

Assuming that the jump intensity matrix is of the form $\varepsilon \cdot Q$ where $\varepsilon > 0$ is a small parameter – which means that the frequency of the jumps is small – it is possible to obtain an asymptotic expansion of the unnormalized conditional distribution in powers of ε.

5 Asymptotic statistics

Some off-line statistical procedures based on likelihood, have been presented. Whether these statistical procedures actually provide good results, has to be investigated in some asymptotic sense. Two kind of asymptotics are generally considered in the statistics of random processes, see Kutoyants [5]

- the small noise asymptotics, where the noise covariances are of order $\sqrt{\varepsilon}$, and ε is sent to zero,
- the long-time asymptotics, where the observation horizon T is sent to infinity.

This section is devoted to presenting a recent result on the consistency of the MLE in the small noise asymptotics, see James-LeGland [4].

Statistical model On the canonical space $\Omega = C([0,T]\,;\, \mathbf{R}^{m+d})$ are given

- a pair of stochastic processes $\{X_t\,,\, 0 \le t \le T\}$ and $\{Y_t\,,\, 0 \le t \le T\}$ taking values in $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively,
- for each $\varepsilon > 0$, a family $\mathcal{M}^\varepsilon = \{P_{\theta,\varepsilon}\,,\, \theta \in \Theta\}$ of probability measures,

such that under $P_{\theta,\varepsilon}$

$$\begin{aligned} dX_t &= b_\theta(X_t)\, dt + \sqrt{\varepsilon}\, dW^{\theta,\varepsilon}_t \\ dY_t &= h_\theta(X_t)\, dt + \sqrt{\varepsilon}\, dV^{\theta,\varepsilon}_t \end{aligned} \qquad (17)$$

where $\{W^{\theta,\varepsilon}_t\,,\, 0 \le t \le T\}$ and $\{V^{\theta,\varepsilon}_t\,,\, 0 \le t \le T\}$ are independent Wiener processes of appropriate dimension, with covariance matrix I, and the random variable X_0 is independent of the Wiener processes, with probability distribution $p^{\theta,\varepsilon}_0(x)\, dx$. It is assumed that the initial density is of the form

$$p^{\theta,\varepsilon}_0(x) = C_{\theta,\varepsilon} \cdot \exp\{-\frac{1}{\varepsilon} S^\theta_0(x)\}\ ,$$

where the function S^θ_0 has an unique minimizer $\overline{x}^\theta_0$.

Limiting deterministic system For any $\theta \in \Theta$, consider the following deterministic differential system

$$(\Sigma^\theta) \qquad \begin{cases} \dot{x}^\theta_t = b_\theta(x^\theta_t)\ , & x^\theta_0 = \overline{x}^\theta_0 \\ \dot{y}^\theta_t = h_\theta(x^\theta_t)\ , & y^\theta_0 = 0 \end{cases}$$

which is obtained from (17) by sending ε to zero. This defines a family $\mathcal{M}^0 = \{(\Sigma^\theta)\,,\, \theta \in \Theta\}$ of deterministic differential systems.

Actually, the following convergence in probability of the experiments holds

$$P_{\theta,\varepsilon}(\sup_{0\le t\le T} |Y_t - y_t^\theta| > \delta) \xrightarrow{\varepsilon\downarrow 0} 0 .$$

Deterministic parameter estimation Assume that a trajectory $\{y_t^\alpha , t \geq 0\}$ is observed, which is actually the output of some deterministic differential system (Σ^α) in the model $\mathcal{M}^0$, for some unknown α. The problem is to estimate the parameter $\theta \in \Theta$, based on observations $\{y_t^\alpha , t \geq 0\}$.

The model $\mathcal{M}^0$ is said *identifiable* on $[0,T]$, if for all $\theta' \neq \theta$, there exists $t \in [0,T]$ such that $y_t^{\theta'} \neq y_t^\theta$, i.e. different values of the parameter give different output trajectories. In other words, the mapping $\theta \mapsto \{y_t^\theta , 0 \leq t \leq T\}$ is injective. The deterministic parameter estimation problem consists of inverting this mapping. This can be expressed in terms of the following variational problem.

Introduce the cost functional

$$J_\alpha^\theta(\xi,t) = S_0^\theta(\xi_0) + \tfrac{1}{2}\int_0^t |\dot{\xi}_s - b_\theta(\xi_s)|^2\, ds + \tfrac{1}{2}\int_0^t |\dot{y}_s^\alpha - h_\theta(\xi_s)|^2\, ds - \tfrac{1}{2}\int_0^t |\dot{y}_s^\alpha|^2\, ds ,$$

for absolutely continuous $\xi \in C([0,T]\,;\mathbf{R}^m)$, and the following least-squares functional

$$\ell_\alpha[\theta] = \inf\{J_\alpha^\theta(\xi,T) \,:\, \xi \in C([0,T]\,;\mathbf{R}^m)\} .$$

The deterministic parameter estimate (DPE) is defined by

$$M_\alpha = \operatorname*{argmin}_{\theta\in\Theta} \ell_\alpha[\theta] .$$

The following consistency result can be proved, which relies on PDE techniques for large deviations (*vanishing viscosity* theorem) and on the convergence in probability of the experiments.

Theorem 5.1 *For all $\alpha \in \Theta$, if the deterministic model $\mathcal{M}^0$ is identifiable, then any MLE sequence $\{\widehat{\theta}^\varepsilon , \varepsilon > 0\}$ converges in $P_{\alpha,\varepsilon}$-probability to the "true" parameter : for all $\delta > 0$*

$$P_{\alpha,\varepsilon}(|\widehat{\theta}^\varepsilon - \alpha| > \delta) \xrightarrow{\varepsilon\downarrow 0} 0 .$$

Another issue is the rate of convergence of the MLE sequence to the true value of the parameter. The solution to this question relies on proving a local asymptotic normality (LAN) result. This is currently under investigation.

Other problems should also be considered, including : large time asymptotics, recursive (on-line) estimation, and adaptive filtering.

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CONSISTENT PARAMETER ESTIMATION FOR PARTIALLY OBSERVED DIFFUSIONS WITH SMALL NOISE *

Estimation consistante de paramètres
pour les processus de diffusion partiellement observés

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Abstract

In this paper we provide a consistency result for the MLE for partially observed diffusion processes with small noise intensities. We prove that if the underlying deterministic system enjoys an identifiability property, then any MLE is close to the true parameter if the noise intensities are small enough. The proof uses large deviations limits obtained by PDE vanishing viscosity methods. A deterministic method of parameter estimation is formulated. We also specialize our results to a binary detection problem, and compare deterministic and stochastic notions of identifiability.

Key words: Parameter estimation, nonlinear filtering, large deviations.

1980 subject classifications: 62F12, 93E10, 93E11, 60F10

Résumé

On démontre la consistance du maximum de vraisemblance pour l'estimation de paramètres dans les processus de diffusion partiellement observés, dans le cas de petits bruits. Si le système déterministe sous–jacent est *identifiable*, alors tout estimateur du maximum de vraisemblance est proche de la vraie valeur du paramètre inconnu, pourvu que les bruits soient assez petits. La démonstration utilise des résultats de grandes déviations, qui sont obtenus par des techniques d'EDP (*vanishing viscosity*). On applique ce résultat à un problème de détection séquentielle, et on compare les notions déterministe et stochastique d'identifiabilité.

Mots–Clés: Estimation de paramètres, filtrage non–linéaire, grandes déviations.

Contents

1 Introduction

In this paper we provide a consistency result for the Maximum Likelihood Estimator (MLE) for partially observed diffusions with small noise.

The problem of computing the MLE for partially observed diffusions has received recent attention. Dembo and Zeitouni [7] have investigated the EM algorithm, and Campillo and Le Gland [2] have compared this algorithm with a direct maximization approach. Of course, the goal of such efforts is to compute a good estimate of the unknown parameter. The success or otherwise of such algorithms depends on whether the MLE itself is a good approximation to the unknown parameter. The purpose of this paper is to address this question of consistency when the diffusion and observation noise intensities are "small".

Our result was in part inspired by some large deviations limit results for nonlinear filtering in Hijab [11], James and Baras [12], James [13]. The theory of large deviations for diffusions with small noise is presented in Freidlin and Wentzell [10]. We exploit the fact that, on finite time intervals, the diffusion X with observations Y are "close" to a deterministic process x^α with observations y^α. We formulate a deterministic method of parameter estimation for this deterministic process.

We prove that if the underlying deterministic system is *identifiable* and if α is the true parameter, then any MLE $\hat{\theta}^\varepsilon$ is close to α if $\varepsilon > 0$ is small enough. Our proof uses PDE vanishing viscosity methods and Laplace's asymptotic method.

As an application of our results, we study a binary sequential detection problem, discussed in Baras and La Vigna [1], when the noise intensities are small. Deterministic and stochastic notions of identifiability are compared in the context of threshold decision policies.

2 Maximum Likelihood Estimation

On a measurable space $(\Omega, \mathcal{F})$ we consider

- for each $\varepsilon > 0$, a family $\mathcal{M}^\varepsilon = \{P_{\theta,\varepsilon}\, , \theta \in \Theta\}$ of probability measures,
- a pair of stochastic processes $X \equiv \{X_t\, , 0 \le t \le T\}$ and $Y \equiv \{Y_t\, , 0 \le t \le T\}$ taking values in $\mathbf{R}^m$ and $\mathbf{R}^d$ respectively,

such that under $P_{\theta,\varepsilon}$

$$dX_t = b_\theta(X_t)\, dt + dW_t^{\theta,\varepsilon}\ , \quad X_0 \sim p_0^{\theta,\varepsilon}(x)\, dx\ ,$$

$$dY_t = h_\theta(X_t)\, dt + dV_t^{\theta,\varepsilon}\ , \quad Y_0 = 0\ ,$$

where $\{W_t^{\theta,\varepsilon}\, , 0 \le t \le T\}$ and $\{V_t^{\theta,\varepsilon}\, , 0 \le t \le T\}$ are independent Wiener processes, with covariance matrices εI_m and εI_d respectively, and X_0 is a random variable independent of the Wiener processes, with density of the form

$$p_0^{\theta,\varepsilon}(x) \triangleq C_{\theta,\varepsilon}\ \exp\{-\frac{1}{\varepsilon} S_0^\theta(x)\}\ . \tag{2.1}$$

The set of parameters $\Theta \subset \mathbf{R}^p$ is compact, and the coefficients satisfy the following hypotheses

(i) for all $\theta \in \Theta$, $b_\theta \in C_b^1(\mathbf{R}^m, \mathbf{R}^m)$, and $h_\theta \in C_b^2(\mathbf{R}^m, \mathbf{R}^d)$,

(ii) for all $\theta \in \Theta$, S_0^θ is convex, locally Lipschitz continuous, and for some $\bar{x}_0^\theta \in \mathbf{R}^m$, $S_0^\theta(\bar{x}_0^\theta) = 0$, $S_0^\theta(x) > 0$ if $x \ne \bar{x}_0^\theta$. Assume also

$$C_1 + C_1|x|^2 \ge S_0^\theta(x) \ge C_2|x| - C_2'\ ,$$

for all $x \in \mathbf{R}^n$, $\theta \in \Theta$.

Further, the functions b_θ, h_θ and S_0^θ depend continuously on the parameter θ in the sense that

(iii) for each $\delta > 0$, $R > 0$, there exists $\gamma > 0$ such that $|\theta' - \theta| < \gamma$ implies

$$\sup_{x \in \mathbf{R}^m} |b_{\theta'}(x) - b_\theta(x)| < \delta\ , \quad \sup_{x \in \mathbf{R}^m} |h_{\theta'}(x) - h_\theta(x)| < \delta\ ,$$

$$\sup_{x \in B(0,R)} |S_0^{\theta'}(x) - S_0^\theta(x)| < \delta\ .$$

There is no loss in generality in assuming that Ω is the *canonical space* $C([0,T]\,;\,\mathbf{R}^{m+d})$, in which case X and Y are the *canonical processes* on $C([0,T]\,;\,\mathbf{R}^m)$ and $C([0,T]\,;\,\mathbf{R}^d)$ respectively, and $P_{\theta,\varepsilon}$ is the probability law of (X,Y).

It is assumed that only Y is observed. Let $\mathcal{Y}_T$ denote the σ-algebra generated by the process Y on $C([0,T]\,;\,\mathbf{R}^d)$. The probability measures in $\mathcal{M}^\varepsilon$ are mutually absolutely continuous, and the *log-likelihood function* for estimating the parameter θ in the statistical model $\mathcal{M}^\varepsilon$ given $\mathcal{Y}_T$, can be expressed (note the minus sign) as

$$-\ell^\varepsilon(\theta) = \varepsilon \log \mathbf{E}^\dagger_{\theta,\varepsilon}(Z^{\theta,\varepsilon} \mid \mathcal{Y}_T) \, .$$

Here $P^\dagger_{\theta,\varepsilon}$ is a probability measure equivalent to $P_{\theta,\varepsilon}$, with Radon–Nikodym derivative

$$Z^{\theta,\varepsilon} \triangleq \frac{dP_{\theta,\varepsilon}}{dP^\dagger_{\theta,\varepsilon}} = \exp \frac{1}{\varepsilon} \left\{ \int_0^T h^*_\theta(X_s)\, dY_s - \tfrac{1}{2} \int_0^T |h_\theta(X_s)|^2\, ds \right\} \, ,$$

so that under $P^\dagger_{\theta,\varepsilon}$

$$dX_t = b_\theta(X_t)\, dt + dW^{\theta,\varepsilon}_t \, , \qquad X_0 \sim p^{\theta,\varepsilon}_0(x)\, dx \, ,$$

where $\{W^{\theta,\varepsilon}_t\, , \, t \geq 0\}$ and $\{Y_t\, , \, t \geq 0\}$ are independent Wiener processes, with covariance matrices εI_m and εI_d respectively, and the random variable X_0 is independent of the Wiener processes, see [2].

The *maximum likelihood estimate* (MLE) of the parameter θ in the statistical model $\mathcal{M}^\varepsilon$, is defined on the canonical space $C([0,T]\,;\,\mathbf{R}^d)$ by

$$\widehat{\theta}^\varepsilon \in \operatorname*{argmin}_{\theta \in \Theta} \ell^\varepsilon(\theta) \, .$$

The likelihood function can be computed through the solution of the Zakai equation

$$dp^{\theta,\varepsilon}(x,t) = [L^*_{\theta,\varepsilon} p^{\theta,\varepsilon}](x,t)\, dt + \frac{1}{\varepsilon} h^*_\theta(x) p^{\theta,\varepsilon}(x,t)\, dY_t \, , \tag{2.2}$$

where $L^*_{\theta,\varepsilon}$ is the adjoint operator of the infinitesimal generator $L_{\theta,\varepsilon}$ of the diffusion process X under the probability measure $P_{\theta,\varepsilon}$

$$L_{\theta,\varepsilon} \triangleq \tfrac{1}{2}\varepsilon \sum_{i,j=1}^m \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^m b^i_\theta \frac{\partial}{\partial x_i} \, .$$

Indeed

$$\ell^\varepsilon(\theta) = -\varepsilon \log \int_{\mathbf{R}^m} p^{\theta,\varepsilon}(x,T)\, dx \, . \tag{2.3}$$

The filtering problem is discussed in detail in Liptser and Shiryayev [15]. The following lemma is proved in the Appendix.

Lemma 2.1 *The log–likelihood function $-\ell^\varepsilon(\theta)$ depends continuously on the parameter $\theta \in \Theta$ a.s.*

Let now θ be fixed. When $\varepsilon \downarrow 0$, the following weak convergence result holds on $C([0,T]\,;\,\mathbf{R}^{m+d})$:

$$P_{\theta,\varepsilon} \overset{\varepsilon\downarrow 0}{\Longrightarrow} \delta_{(x^\theta,y^\theta)} \ ,$$

where (x^θ, y^θ) is given by the deterministic differential system

$$(\Sigma^\theta) \quad \begin{cases} \dot{x}_t^\theta = b_\theta(x_t^\theta) \ , & x_0^\theta = \bar{x}_0^\theta \\ \dot{y}_t^\theta = h_\theta(x_t^\theta) \ , & y_0 = 0. \end{cases}$$

In particular, for all $\theta \in \Theta$, $\delta > 0$

$$P_{\theta,\varepsilon}(\sup_{0 \le t \le T} |Y_t - y_t^\theta| > \delta) \xrightarrow{\varepsilon\downarrow 0} 0 \ , \tag{2.4}$$

see Freidlin and Wentzell [10].

Remark 2.2 As long as $\varepsilon > 0$, the probability measures in $\mathcal{M}^\varepsilon$ are mutually absolutely continuous, which allows us to define the log–likelihood function $-\ell^\varepsilon(\theta)$. On the other hand, asymptotically when $\varepsilon \downarrow 0$, these probability measures look more and more mutually singular, which, together with an identifiability property of the underlying deterministic system, indicates that the MLE may be consistent. Actually, this result will be proved below.

The purpose of the next Section is to consider the problem of estimating the unknown parameter θ in the deterministic model $\mathcal{M}^0 = \{(\Sigma^\theta)\, ,\, \theta \in \Theta\}$.

3 Deterministic Parameter Estimation

Consider the family $\mathcal{M}^0 = \{(\Sigma^\theta), \theta \in \Theta\}$ of deterministic differential systems

$$(\Sigma^\theta) \quad \begin{cases} \dot{x}_t^\theta = b_\theta(x_t^\theta), & x_0^\theta = \bar{x}_0^\theta \\ \dot{y}_t^\theta = h_\theta(x_t^\theta), & y_0 = 0. \end{cases} \tag{3.1}$$

Note that for all $\theta \in \Theta$, (Σ^θ) describes the weak limit as $\varepsilon \downarrow 0$ of the family of probability measures $\{P_{\theta,\varepsilon}, \varepsilon > 0\}$.

The problem is to estimate the unknown parameter θ on the basis of an observation record, which is supposed to be the output of some deterministic differential systems in $\mathcal{M}^0$. Introduce the following definition:

Definition 3.1 *The model $\mathcal{M}^0$ is* identifiable *on $[0,T]$ if for all $\theta' \neq \theta$ in Θ, there exists $t \in [0,T]$ such that*

$$y_t^{\theta'} \neq y_t^\theta .$$

In other words, the mapping $\theta \mapsto \{y_t^\theta, 0 \leq t \leq T\}$ is injective. The deterministic parameter estimation problem consists of inverting this mapping. This can be expressed in terms of the following variational problem.

Define the following functional on $C([0,T]; \mathbf{R}^m)$

$$\begin{aligned} J_\alpha^\theta(\xi, t) \triangleq\ & S_0^\theta(\xi_0) + \tfrac{1}{2}\int_0^t |\dot{\xi}_s - b_\theta(\xi_s)|^2\, ds \\ & + \tfrac{1}{2}\int_0^t |\dot{y}_s^\alpha - h_\theta(\xi_s)|^2\, ds - \tfrac{1}{2}\int_0^t |\dot{y}_s^\alpha|^2\, ds , \end{aligned} \tag{3.2}$$

if ξ is absolutely continuous, $J_\alpha^\theta(\xi, t) = +\infty$ otherwise. For all $x \in \mathbf{R}^m$ set

$$W_\alpha^\theta(x,t) \triangleq \inf\left\{ J_\alpha^\theta(\xi, t) \,:\, \xi_t = x \right\} . \tag{3.3}$$

The value function $W_\alpha^\theta(x,t)$ is continuous in (x,t) and is the unique *viscosity solution* of the Hamilton–Jacobi equation [12]

$$\frac{\partial}{\partial t} W_\alpha^\theta(x,t) + H_\alpha^\theta(x,t,DW_\alpha^\theta(x,t)) = 0 , \quad W_\alpha^\theta(x,0) = S_0^\theta(x) , \tag{3.4}$$

where the Hamiltonian $H_\alpha^\theta(x,t,\lambda)$ is defined by

$$\begin{aligned} H_\alpha^\theta(x,t,\lambda) &\triangleq \max_{u \in \mathbf{R}^m} \left\{ \lambda^*(b_\theta(x) + u) - \tfrac{1}{2}|u|^2 \right\} - \tfrac{1}{2}|\dot{y}_t^\alpha - h_\theta(x)|^2 + \tfrac{1}{2}|\dot{y}_t^\alpha|^2 \\ &= b_\theta^*(x)\lambda + \tfrac{1}{2}|\lambda|^2 + h_\theta^*(x)\dot{y}_t^\alpha - \tfrac{1}{2}|h_\theta(x)|^2 . \end{aligned} \tag{3.5}$$

For definitions and an introduction to viscosity solutions of Hamilton–Jacobi equations, the reader is refered to Crandall and Lions [3], Crandall, Evans and Lions [5].

Consider the following functional, defined on Θ by

$$\ell_\alpha(\theta) \triangleq \inf_{x\in\mathbf{R}^m} W_\alpha^\theta(x,T) = \inf\left\{J_\alpha^\theta(\xi,T) \,:\, \xi\in C([0,T];\mathbf{R}^m)\right\} . \tag{3.6}$$

A *deterministic estimate* (DPE) of the unknown parameter θ in the model $\mathcal{M}^0$ on the basis of the observation record $\{y_t^\alpha\,,\, 0\leq t\leq T\}$ is defined by

$$\widehat{\theta}_\alpha \in M_\alpha \triangleq \operatorname*{argmin}_{\theta\in\Theta} \ell_\alpha(\theta) . \tag{3.7}$$

The main result of this section is the following:

Theorem 3.2 *If the model $\mathcal{M}^0$ is identifiable, then for all $\alpha\in\Theta$*

$$M_\alpha = \{\alpha\} .$$

Thus, under the identifiability hypothesis, the DPE is uniquely defined and the unknown parameter can, in principle, be computed exactly from (3.4), (3.6), (3.7). Before proving Theorem 3.2, we give a lemma which ensures that $\operatorname*{argmin}_{\theta\in\Theta}\ell_\alpha(\theta)\neq\emptyset$, and also provides useful estimates.

Lemma 3.3 *For all $\alpha\in\Theta$*

(i) there are constants $C>0$, $C'>0$ such that, for all $x\in\mathbf{R}^m$, $\theta\in\Theta$

$$C_1|x|^2 + C \geq W_\alpha^\theta(x,T) \geq C|x| - C' ,$$

(ii) for all $R>0$, $\delta>0$ there exists $\gamma>0$ such that $|\theta'-\theta|<\gamma$ implies

$$\sup_{x\in B(0,R)} |W_\alpha^{\theta'}(x,T) - W_\alpha^\theta(x,T)| < \delta ,$$

(iii) the mapping $\theta\longmapsto \ell_\alpha(\theta)$ is continuous.

PROOF. In the sequel, every constant independent of $\theta,\alpha\in\Theta$ and $(x,t)\in\mathbf{R}^m\times[0,T]$ will be denoted by C or C'. For any absolutely continuous function $\xi\in C([0,T];\mathbf{R}^m)$ and any $\Delta>0$, we have

$$|\xi_t|^2 \leq |\xi_s|^2 + \frac{1}{\Delta}\int_s^t |\xi_\tau|^2\,d\tau + \Delta\int_s^t |\dot{\xi}_\tau|^2\,d\tau ,$$

and by Gronwall's lemma,

$$|\xi_t|^2 \le \left(|\xi_s|^2 + \Delta \int_s^t |\dot\xi_\tau|^2 d\tau\right) \exp\{(t-s)/\Delta\} . \tag{3.8}$$

Since $\sup_{\theta\in\Theta} \sup_{x\in\mathbf{R}^m} |h_\theta(x)| \le C$ it follows that, for all $\alpha \in \Theta$

$$\tfrac{1}{2} \int_0^T |\dot y_s^\alpha|^2 \, ds \le C ,$$

and hence for all $\alpha, \theta \in \Theta$

$$W_\alpha^\theta(x,t) \ge -C , \qquad (x,t) \in \mathbf{R}^m \times [0,T] .$$

Let $L_\alpha^\theta(\dot\xi, \xi, t)$ denote the Lagrangian in (3.2). It is easy to prove the following estimates

$$\tfrac{1}{4} \int_s^t |\dot\xi_\tau|^2 \, d\tau \le \tfrac{1}{2} \int_s^t |\dot\xi_\tau - b_\theta(\xi_\tau)|^2 \, d\tau + \tfrac{1}{2} \int_s^t |b_\theta(\xi_\tau)|^2 \, d\tau \le \int_s^t L_\alpha^\theta(\dot\xi_\tau, \xi_\tau, \tau) d\tau + C ,$$

$$\tfrac{1}{2} \int_s^t |\dot\xi_\tau - b_\theta(\xi_\tau)|^2 \, d\tau \le \int_s^t |\dot\xi_\tau|^2 \, d\tau + \int_s^t |b_\theta(\xi_\tau)|^2 \, d\tau \le \int_s^t |\dot\xi_\tau|^2 \, d\tau + C .$$

In particular

$$\tfrac{1}{4} \int_0^t |\dot\xi_s|^2 \, ds - C \le J_\alpha^\theta(\xi, t) \le S_0^\theta(\xi_0) + \int_0^t |\dot\xi_s|^2 \, ds + C .$$

Proof of (i): Setting $\xi \equiv x$ on $[0,T]$, gives for $0 \le t \le T$

$$W_\alpha^\theta(x,t) \le J_\alpha^\theta(\xi,t) \le S_0^\theta(x) + C \le C_1|x|^2 + C .$$

Choose $\Delta > 0$ such that $N = T/\Delta$ is an integer and $4eC_1\Delta \le \frac{1}{2}$. For $n = 1, \ldots, N$ the Dynamic Programming principle implies

$$W_\alpha^\theta(z, n\Delta) = \inf_\xi \left\{ W(\xi_{(n-1)\Delta}, (n-1)\Delta) + \int_{(n-1)\Delta}^{n\Delta} L_\alpha^\theta(\dot\xi_s, \xi_s, s) ds \; : \; \xi_{n\Delta} = z \right\} .$$

Given $\delta > 0$, recursively select $\xi^n \in C([0,T]\,; \mathbf{R}^m)$ for $n = N, \ldots, 1$ as follows: $\xi_{N\Delta}^N = x$, $\xi_{n\Delta}^{n-1} = \xi_{n\Delta}^n$ and

$$\begin{aligned} W_\alpha^\theta(\xi_{(n-1)\Delta}^n, (n-1)\Delta) + \int_{(n-1)\Delta}^{n\Delta} L_\alpha^\theta(\dot\xi_s^n, \xi_s^n, s) ds \; &\le \; W_\alpha^\theta(\xi_{n\Delta}^n, n\Delta) + \frac{\delta}{N} \\ &\le C_1|\xi_{n\Delta}^n|^2 + C + \frac{\delta}{N} . \end{aligned} \tag{3.9}$$

Then

$$\tfrac{1}{4} \int_{(n-1)\Delta}^{n\Delta} |\dot\xi_s^n|^2 \, ds \le C_1|\xi_{n\Delta}^n|^2 + C + \frac{\delta}{N} ,$$

and from (3.8)

$$|\xi^n_{n\Delta}|^2 \le \left(|\xi^n_{(n-1)\Delta}|^2 + \Delta \int_{(n-1)\Delta}^{n\Delta} |\dot{\xi}^n_\tau|^2 d\tau\right) e \le e\, |\xi^n_{(n-1)\Delta}|^2 + \tfrac{1}{2}|\xi^n_{n\Delta}|^2 + \tfrac{1}{2}(C + \frac{\delta}{N})/C_1 ,$$

which implies

$$|\xi^n_{n\Delta}|^2 \le 2e\, |\xi^n_{(n-1)\Delta}|^2 + (C + \frac{\delta}{N})/C_1 . \tag{3.10}$$

Define $\xi^\theta \in C([0,T]\,;\mathbf{R}^m)$ by $\xi^\theta_t = \xi^n_t$ for $t \in [(n-1)\Delta, n\Delta]$, $n = 1, \ldots, N$. Then $\xi^\theta_T = x$ and by iterating (3.10) we obtain

$$|x|^2 \le C^N\, |\xi^\theta_0|^2 + C^N .$$

Now also, by iterating (3.9)

$$J^\theta_\alpha(\xi^\theta, T) \le W^\theta_\alpha(x,T) + \delta , \tag{3.11}$$

and consequently

$$W^\theta_\alpha(x,T) \ge J^\theta_\alpha(\xi^\theta, T) - \delta \ge S^\theta_0(\xi^\theta_0) - C \ge C|x| - C' ,$$

which proves (i).

Proof of (ii): Let $R > 0$, $\delta > 0$ and $x \in B(0,R)$. Choose ξ^θ as in (3.11). Then, from the above estimates,

$$\int_0^T |\dot{\xi}^\theta_s|^2\, ds \le C_R .$$

Using (3.8) we deduce that if $x \in B(0,R)$, then there exists $R' > 0$ such that $\xi^\theta_0 \in B(0,R')$ for all $\theta \in \Theta$. Therefore

$$\begin{aligned}
& W^{\theta'}_\alpha(x,T) - W^\theta_\alpha(x,T) \\
\le\ & J^{\theta'}_\alpha(\xi^\theta, T) - J^\theta_\alpha(\xi^\theta, T) + \tfrac{1}{4}\delta \\
=\ & S^{\theta'}_0(\xi^\theta_0) - S^\theta_0(\xi^\theta_0) + \tfrac{1}{2}\int_0^T |\dot{\xi}^\theta_s - b_{\theta'}(\xi^\theta_s)|^2\, ds - \tfrac{1}{2}\int_0^T |\dot{\xi}^\theta_s - b_\theta(\xi^\theta_s)|^2\, ds \\
& + \tfrac{1}{2}\int_0^T |\dot{y}^\alpha_s - h_{\theta'}(\xi^\theta_s)|^2\, ds - \tfrac{1}{2}\int_0^T |\dot{y}^\alpha_s - h_\theta(\xi^\theta_s)|^2\, ds + \tfrac{1}{4}\delta .
\end{aligned}$$

Now, if $|\theta' - \theta|$ is small enough

$$|S^{\theta'}_0(\xi^\theta_0) - S^\theta_0(\xi^\theta_0)| < \tfrac{1}{4}\delta ,$$

$$\tfrac{1}{2}\left|\int_0^T |\dot{y}^\alpha_s - h_{\theta'}(\xi^\theta_s)|^2\, ds - \int_0^T |\dot{y}^\alpha_s - h_\theta(\xi^\theta_s)|^2\, ds\right| < \tfrac{1}{4}\delta .$$

Also

$$\tfrac{1}{2}\left|\int_0^T |\dot{\xi}_s^\theta - b_{\theta'}(\xi_s^\theta)|^2\,ds - \int_0^T |\dot{\xi}_s^\theta - b_\theta(\xi_s^\theta)|^2\,ds\right|$$

$$\le \left\{\int_0^T |\dot{\xi}_s^\theta|^2\,ds\right\}^{1/2} \left\{\int_0^T |b_{\theta'}(\xi_s^\theta) - b_\theta(\xi_s^\theta)|^2\,ds\right\}^{1/2}$$

$$+\tfrac{1}{2}\int_0^T |b_{\theta'}(\xi_s^\theta) - b_\theta(\xi_s^\theta)|\,|b_{\theta'}(\xi_s^\theta) + b_\theta(\xi_s^\theta)|\,ds < \tfrac{1}{4}\delta\ ,$$

if $|\theta' - \theta|$ is small enough. Hence, there exists $\gamma > 0$ such that $|\theta' - \theta| < \gamma$ implies

$$W_\alpha^{\theta'}(x,T) - W_\alpha^\theta(x,T) < \delta\ .$$

Reversing the role of θ' and θ proves (ii).

Finally, (iii) follows from (i)–(ii) and Lemma A.2. □

PROOF OF THEOREM 3.2. From (3.2), (3.3) and (3.6) we have

$$J_\alpha^\theta(\xi,T) \ge c_\alpha \triangleq -\tfrac{1}{2}\int_0^T |\dot{y}_s^\alpha|^2\,ds\ ,$$

for all $\theta \in \Theta$ and $\xi \in C([0,T]\,;\,\mathbf{R}^m)$, so that $\ell_\alpha(\theta) \ge c_\alpha$. From (3.1) we have

$$J_\alpha^\alpha(x^\alpha,T) = c_\alpha\ ,$$

so that for all $\theta \in \Theta$, $\ell_\alpha(\alpha) = c_\alpha \le \ell_\alpha(\theta)$. Therefore $\alpha \in M_\alpha$.

Assume that $\widehat{\theta} \in M_\alpha$. Then $\ell_\alpha(\widehat{\theta}) = \ell_\alpha(\alpha)$ and

$$\ell_\alpha(\widehat{\theta}) = \inf\{J_\alpha^{\widehat{\theta}}(\xi,T)\ :\ \xi \in C([0,T]\,;\,\mathbf{R}^m)\} = J_\alpha^{\widehat{\theta}}(\widehat{\xi},T)\ ,$$

for some $\widehat{\xi} \in C([0,T]\,;\,\mathbf{R}^m)$, since $J_\alpha^{\widehat{\theta}}(\cdot,T)$ is lower semi-continuous. Then from (3.2)

$$(i)\qquad S_0^{\widehat{\theta}}(\widehat{\xi}_0) = 0\ ,$$

$$(ii)\qquad \dot{\widehat{\xi}}_s = b_{\widehat{\theta}}(\widehat{\xi}_s)\ ,\quad 0 \le s \le T\ ,$$

$$(iii)\qquad \dot{y}_s^\alpha = h_{\widehat{\theta}}(\widehat{\xi}_s)\ ,\quad 0 \le s \le T\ .$$

From (i) $\widehat{\xi}_0 = \bar{x}_0^{\widehat{\theta}}$, and therefore by (ii) and (3.1) $\widehat{\xi}_s = x_s^{\widehat{\theta}}$, $0 \le s \le T$. Then (iii) and (3.1) imply

$$\dot{y}_s^{\widehat{\theta}} = h_{\widehat{\theta}}(x_s^{\widehat{\theta}}) = \dot{y}_s^\alpha\ ,\quad 0 \le s \le T\ .$$

Now since the model $\mathcal{M}^0$ is identifiable, this equality forces $\widehat{\theta} = \alpha$, which proves the theorem. □

Remark 3.4 The notion of identifiability is reminiscent of a notion of observability for nonlinear systems, which also has a variational characterization, see James [13] [14].

4 Consistency Result for MLE

The main result of this paper is the following:

Theorem 4.1 *For all $\alpha \in \Theta$*

(i) any MLE sequence $\{\widehat{\theta}^\varepsilon, \varepsilon > 0\}$ converges in $P_{\alpha,\varepsilon}$-probability to the deterministic set M_α: for all $\delta > 0$

$$P_{\alpha,\varepsilon}(d(\widehat{\theta}^\varepsilon, M_\alpha) > \delta) \xrightarrow{\varepsilon\downarrow 0} 0 \ ,$$

(ii) if the deterministic model $\mathcal{M}^0$ is identifiable, then any MLE sequence $\{\widehat{\theta}^\varepsilon, \varepsilon > 0\}$ converges in $P_{\alpha,\varepsilon}$-probability to the "true" parameter: for all $\delta > 0$

$$P_{\alpha,\varepsilon}(|\widehat{\theta}^\varepsilon - \alpha| > \delta) \xrightarrow{\varepsilon\downarrow 0} 0 \ .$$

The proof of this theorem depends on a technical extension of large deviations limit results for nonlinear filtering contained in James and Baras [12], James [13]. We need to show that certain limits are uniform in the parameter $\theta \in \Theta$. The key technical lemma is the following:

Lemma 4.2 *The sequence $\{\ell^\varepsilon(\theta), \varepsilon > 0\}$ converges in $P_{\alpha,\varepsilon}$-probability uniformly in $\theta \in \Theta$ to $\ell_\alpha(\theta)$: for all $\delta > 0$*

$$P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} |\ell^\varepsilon(\theta) - \ell_\alpha(\theta)| > \delta) \xrightarrow{\varepsilon\downarrow 0} 0 \ .$$

We next prove Theorem 4.1 using Lemma 4.2; the remainder of this section is concerned with proving Lemma 4.2.

PROOF OF THEOREM 4.1. By Lemma A.1 for all $\delta > 0$ there exists $\gamma > 0$ such that

$$\{\sup_{\theta\in\Theta} |\ell^\varepsilon(\theta) - \ell_\alpha(\theta)| < \gamma\} \subset \{d(\widehat{\theta}^\varepsilon, M_\alpha) < \delta\} \ .$$

Therefore, by Lemma 4.2

$$P_{\alpha,\varepsilon}(d(\widehat{\theta}^\varepsilon, M_\alpha) > \delta) \leq P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} |\ell^\varepsilon(\theta) - \ell_\alpha(\theta)| > \gamma) \xrightarrow{\varepsilon\downarrow 0} 0 \ ,$$

which proves (i).

The proof of (ii) follows at once from (i) and Theorem 3.2. □

As in James and Baras [12], James [13], we employ the vanishing viscosity method of Evans and Ishii [8]. We proceed by a logarithmic change of variables used by Fleming and Mitter [9]. Define

$$W^{\theta,\epsilon}(x,t) \triangleq -\epsilon \log p^{\theta,\epsilon}(x,t) \ . \tag{4.1}$$

The $\mathcal{Y}_t$-measurable random variable $W^{\theta,\epsilon}(x,t) + h^*_\theta(x)Y_t$ can be extended to a continuous function defined on the whole canonical space $\Omega_0 \equiv \{\eta \in C([0,T]\,;\,\mathbf{R}^d) \,:\, \eta_0 = 0\}$, which we denote by $u^{\theta,\epsilon}[\eta](x,t)$, see [9] and [12]. For any *fixed* $\eta \in \Omega_0$

$$u^{\theta,\epsilon}[\eta] \in C^{2,1}(\mathbf{R}^m \times [0,T]\,;\,\mathbf{R})$$

is the unique solution of the Hamilton–Jacobi–Bellman equation

$$\begin{gathered} \frac{\partial}{\partial t} u^{\theta,\epsilon}[\eta](x,t) - \tfrac{1}{2}\epsilon \Delta u^{\theta,\epsilon}[\eta](x,t) + H^{\theta,\epsilon}[\eta](x,t,Du^{\theta,\epsilon}[\eta](x,t)) = 0 \\ u^{\theta,\epsilon}[\eta](x,0) = S^\theta_0(x) - \epsilon \log C_{\theta,\epsilon} \end{gathered} \tag{4.2}$$

where the Hamiltonian $H^{\theta,\epsilon}[\eta](x,t,\lambda)$ is defined by

$$\begin{gathered} H^{\theta,\epsilon}[\eta](x,t,\lambda) \triangleq g^*_\theta(x,\eta_t)\lambda + \tfrac{1}{2}|\lambda|^2 - V^{\theta,\epsilon}(x,\eta_t) \ , \\ V^{\theta,\epsilon}(x,\eta) \triangleq V^\theta(x,\eta) + \tfrac{1}{2}\epsilon\eta^* \Delta h_\theta(x) + \epsilon \operatorname{div} g_\theta(x,\eta) \ , \\ V^\theta(x,\eta) \triangleq \tfrac{1}{2}|h_\theta(x)|^2 + b^*_\theta \eta^* D h_\theta(x) - \tfrac{1}{2}(Dh_\theta(x))^* \eta\, \eta^*\, Dh_\theta(x) \ , \\ g_\theta(x,\eta) \triangleq b_\theta(x) - \eta^* D h_\theta(x) \ . \end{gathered} \tag{4.3}$$

Next, for $\eta \in \Omega_0$ let

$$u^\theta[\eta] \in C(\mathbf{R}^m \times [0,T]\,;\,\mathbf{R})$$

denote the unique viscosity solution of the Hamilton–Jacobi equation

$$\frac{\partial}{\partial t} u^\theta[\eta](x,t) + H^\theta[\eta](x,t,Du^\theta[\eta](x,t)) = 0 \ , \quad u^{\theta,\epsilon}[\eta](x,0) = S^\theta_0(x) \tag{4.4}$$

where the Hamiltonian $H^\theta[\eta](x,t,\lambda)$ is defined by

$$H^\theta[\eta](x,t,\lambda) \triangleq g^*_\theta(x,\eta_t)\lambda + \tfrac{1}{2}|\lambda|^2 - V^\theta(x,\eta_t) \ . \tag{4.5}$$

Lemma 4.3 *We have*

$$\lim_{\epsilon \downarrow 0} u^{\theta,\epsilon}[\eta](x,t) = u^\theta[\eta](x,t) \ ,$$

uniformly in $\theta \in \Theta$ and $t \in [0,T]$ and uniformly on compact subsets of $\eta \in \Omega_0$ and $x \in \mathbf{R}^m$.

PROOF. The following estimates are obtained as in James and Baras [12], James [13], using methods introduced in Evans and Ishii [8], Crandall and Lions [4]. Let $R > 0$ and $K \subset \Omega_0$ be compact. Then if $\varepsilon > 0$ is sufficiently small, we have

$$|u^{\theta,\varepsilon}[\eta](x,t)| \leq C$$

$$|Du^{\theta,\varepsilon}[\eta](x,t)| \leq C$$

$$|u^{\theta,\varepsilon}[\eta](x,t) - u^{\theta,\varepsilon}[\eta](x,s)| \leq C(\sqrt{\varepsilon}|t-s|^{\frac{1}{2}} + |t-s|)$$

for some constant $C > 0$ and for all $\theta \in \Theta$, $t \in [0,T]$, $\eta \in K$ and $x \in B(0,R)$. By the Arzela–Ascoli theorem, there is a subsequence $\varepsilon_k \downarrow 0$ such that $u^{\theta,\varepsilon_k}[\eta]$ converges uniformly on $B(0,R) \times [0,T]$ to a continuous function w. This function satisfies the Hamilton–Jacobi equation (4.4), and by uniqueness, $w = u^\theta[\eta]$ (Crandall and Lions [3]). Hence $u^{\theta,\varepsilon}[\eta] \to u^\theta[\eta]$ as $\varepsilon \downarrow 0$.

Now $u^\theta[\eta]$ is a continuous function of $\eta \in K$, $\theta \in \Theta$ (see the proof of Lemma 3.3 (ii)). Using this fact and the uniform estimate above we conclude that the convergence is uniform. □

Now

$$W^{\theta,\varepsilon}(x,t) = u^{\theta,\varepsilon}[Y](x,t) - h_\theta^*(x)Y_t$$

and

$$W_\alpha^\theta(x,t) = u^\theta[y^\alpha](x,t) - h_\theta^*(x)y_t^\alpha \ .$$

Lemma 4.4 *We have*

$$\lim_{\varepsilon\downarrow 0} W^{\theta,\varepsilon}(x,t) = W_\alpha^\theta(x,t)$$

in $P_{\alpha,\varepsilon}$-probability uniformly in $\theta \in \Theta$, $t \in [0,T]$ and uniformly on compact subsets of $x \in \mathbf{R}^m$.

PROOF. Let ρ denote a metric on $C(\mathbf{R}^m \times [0,T], \mathbf{R})$ corresponding to uniform convergence on compact subsets. By (2.4), it is enough to show that for each $\delta > 0$

$$P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} \rho(u^{\theta,\varepsilon}[Y], u^\theta[y^\alpha]) > \delta) \xrightarrow{\varepsilon\downarrow 0} 0 \ .$$

Choose $\beta > 0$ such that $\|\eta - y^\alpha\| < \beta$ implies

$$\sup_{\theta\in\Theta} \rho(u^\theta[\eta], u^\theta[y^\alpha]) < \tfrac{1}{2}\delta \ .$$

From Lemma 4.3, if $\|\eta - y^\alpha\| < \beta$ and $0 < \varepsilon \leq \varepsilon_0$ then

$$\sup_{\theta\in\Theta} \rho(u^{\theta,\varepsilon}[\eta], u^\theta[\eta]) < \tfrac{1}{2}\delta \ .$$

Therefore, if $0 < \varepsilon \leq \varepsilon_0$ then

$$
\begin{aligned}
& P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} \rho(u^{\theta,\varepsilon}[Y], u^{\theta}[y^{\alpha}]) > \delta) \\
\leq\ & P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} \rho(u^{\theta,\varepsilon}[Y], u^{\theta}[Y]) > \tfrac{1}{2}\delta\,;\ \|Y - y^{\alpha}\| < \beta) \\
& + P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} \rho(u^{\theta}[Y], u^{\theta}[y^{\alpha}]) > \tfrac{1}{2}\delta\,;\ \|Y - y^{\alpha}\| < \beta) \\
& + P_{\alpha,\varepsilon}(\|Y - y^{\alpha}\| > \beta) \leq P_{\alpha,\varepsilon}(\|Y - y^{\alpha}\| > \beta) \xrightarrow{\varepsilon\downarrow 0} 0\ ,
\end{aligned}
$$

by (2.4). □

PROOF OF LEMMA 4.2. Recall from (2.3) and (3.6) that

$$
\begin{aligned}
\ell^{\varepsilon}(\theta) &= -\varepsilon \log \int_{\mathbf{R}^m} \exp\left\{-\frac{1}{\varepsilon} W^{\theta,\varepsilon}(x,T)\right\} dx \quad \text{a.s.} \\
\ell_{\alpha}(\theta) &= \inf_{x\in\mathbf{R}^m} W^{\theta}_{\alpha}(x,T)\ .
\end{aligned}
$$

From the proof of Lemma 2.1 we see that

$$
W^{\theta,\varepsilon}(x,T) \geq C|x| - C'\ , \quad \text{a.s.}
$$

for all $\varepsilon > 0$, $\theta \in \Theta$, where C' is random and satisfies the following estimate

$$
C' \leq C_0 \|Y - y^{\alpha}\|^2 + C'_{\alpha}\ .
$$

From Lemma A.3, there exists $\varepsilon_0 > 0$, $\beta > 0$ and $c > 0$ such that $0 < \varepsilon \leq \varepsilon_0$

$$
\sup_{\theta\in\Theta} \rho(W^{\theta,\varepsilon}, W^{\theta}_{\alpha}) < \beta \quad \text{and} \quad \|Y - y^{\alpha}\| < c
$$

implies

$$
\sup_{\theta\in\Theta} |\ell^{\varepsilon}(\theta) - \ell_{\alpha}(\theta)| < \delta\ .
$$

Therefore, for $0 < \varepsilon \leq \varepsilon_0$

$$
\begin{aligned}
& P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} |\ell^{\varepsilon}(\theta) - \ell_{\alpha}(\theta)| > \delta) \\
\leq\ & P_{\alpha,\varepsilon}(\sup_{\theta\in\Theta} \rho(W^{\theta,\varepsilon}, W^{\theta}_{\alpha}) > \beta) + P_{\alpha,\varepsilon}(\|Y - y^{\alpha}\| > c) \xrightarrow{\varepsilon\downarrow 0} 0\ ,
\end{aligned}
$$

by (2.4) and Lemma 4.4. □

5 Binary Sequential Detection

In this section we discuss some aspects of a binary detection problem studied by Baras and LaVigna [1], when the noise intensities are small.

Let $\Theta = \{0, 1\}$ and let X and Y be the signal and the observation processes described in Section 2. For $\varepsilon > 0$ fixed, we consider the two hypotheses H_0 and H_1. Under H_0 the law of (X, Y) is $P_{0,\varepsilon}$, whilst under H_1 the law of (X, Y) is $P_{1,\varepsilon}$. The problem is to determine which hypothesis is true, that is to detect the signal. In this section, $\Omega = C([0, \infty), \mathbf{R}^{m+d})$.

A key technical assumption, essentially an identifiability condition, used in Baras and LaVigna [1] is the following

$$\int_0^\infty |\hat{h}_{0,\varepsilon}(t) - \hat{h}_{1,\varepsilon}(t)|^2 \, dt = \infty \quad \text{a.s.} \tag{5.1}$$

where

$$\hat{h}_{\theta,\varepsilon}(t) \triangleq E_{\theta,\varepsilon}(h_\theta(X_t) \mid \mathcal{Y}_t) ,$$

and

$$\mathcal{Y}_t \triangleq \sigma(Y_s \, , \, 0 \le s \le t) .$$

The deterministic analogue of (5.1) is

$$\int_0^\infty |\dot{y}_t^0 - \dot{y}_t^1|^2 \, dt = \infty . \tag{5.2}$$

Clearly, (5.2) implies that the model $\mathcal{M}^0$ defined by (3.1) is identifiable. In fact, if

$$\sigma \triangleq \inf\{T \ge 0 \, : \, \int_0^T |\dot{y}_t^0 - \dot{y}_t^1|^2 \, dt > 0\} ,$$

then $\mathcal{M}^0$ is identifiable on each interval $[0, T]$ with $T > \sigma$.

The following result is a consequence of Theorem 4.1.

Theorem 5.1 *Assume (5.2) holds and $T > \sigma$. Define the MLE $\hat{\theta}^\varepsilon$ for the interval $[0, T]$. Then, for $\alpha = 0, 1$*

$$P_{\alpha,\varepsilon}(\hat{\theta}^\varepsilon = \alpha) \xrightarrow{\varepsilon \downarrow 0} 1 .$$

In [1], Baras and LaVigna use a threshold decision policy to decide which of the hypotheses is valid. Define the likelihood ratio

$$\Lambda_T^\varepsilon \triangleq \exp \frac{1}{\varepsilon} \left\{ \int_0^T [\hat{h}_{1,\varepsilon}(t) - \hat{h}_{0,\varepsilon}(t)]^* \, dY_t - \tfrac{1}{2} \int_0^T [|\hat{h}_{1,\varepsilon}(t)|^2 - |\hat{h}_{0,\varepsilon}(t)|^2] \, dt \right\} .$$

Note that as $\varepsilon \downarrow 0$,

$$\varepsilon \log \Lambda_T^\varepsilon \asymp \begin{cases} -\frac{1}{2}\int_0^T |\dot{y}_t^1 - \dot{y}_t^0|^2\, dt & \text{under } \mathrm{H}_0 \,, \\ +\frac{1}{2}\int_0^T |\dot{y}_t^0 - \dot{y}_t^1|^2\, dt & \text{under } \mathrm{H}_1 \,. \end{cases}$$

A threshold policy $u^\varepsilon = (\tau^\varepsilon, \delta^\varepsilon)$ consists of a $\{\mathcal{Y}_t\,,\, t \geq 0\}$–stopping time τ^ε and a $\mathcal{Y}_{\tau^\varepsilon}$–measurable $\{0,1\}$–valued random variable δ^ε defined by

$$\tau^\varepsilon \triangleq \inf\{T \geq 0 \,:\, \Lambda_T^\varepsilon \notin (e^{a/\varepsilon}, e^{b/\varepsilon})\}\,,$$

$$\delta^\varepsilon \triangleq \begin{cases} 1 & \text{if } \;\Lambda_{\tau^\varepsilon}^\varepsilon = e^{b/\varepsilon}\,, \\ 0 & \text{if } \;\Lambda_{\tau^\varepsilon}^\varepsilon = e^{a/\varepsilon}\,, \end{cases}$$

for some constants $a < 0 < b$. If $\delta^\varepsilon = 1$ we decide that hypothesis H_1 is valid (i.e. that $\theta = 1$), whilst if $\delta^\varepsilon = 0$ we decide H_0 (i.e. $\theta = 0$). Of course, our decision may be in error. Define an error probability for the policy u^ε

$$e(u^\varepsilon) \triangleq P_{0,\varepsilon}(\delta^\varepsilon = 1) + P_{1,\varepsilon}(\delta^\varepsilon = 0)\,.$$

Theorem 5.2 *If (5.1) holds, then*

$$e(u^\varepsilon) \xrightarrow{\varepsilon \downarrow 0} 0\,.$$

PROOF. Under assumption (5.1), Baras and LaVigna [1] prove that

$$\tau^\varepsilon < \infty \quad \text{a.s}$$

and

$$P_{0,\varepsilon}(\delta^\varepsilon = 1) = \frac{1 - e^{a/\varepsilon}}{e^{b/\varepsilon} - e^{a/\varepsilon}}\,, \quad P_{1,\varepsilon}(\delta^\varepsilon = 0) = \frac{e^{a/\varepsilon}(e^{b/\varepsilon} - 1)}{e^{b/\varepsilon} - e^{a/\varepsilon}}\,.$$

Since $a < 0 < b$, the conclusion follows. □

Thus, assuming (5.1), the probability of making an incorrect decision converges to zero as $\varepsilon \downarrow 0$, and so (5.1) can be viewed as an identifiability criterion for the statistical model $\mathcal{M}^\varepsilon = \{P_{0,\varepsilon}\,,\, P_{1,\varepsilon}\}$.

We can define a deterministic threshold policy $u = (\tau, \delta)$ as follows. Define

$$F_T = \frac{1}{2}\int_0^T |\dot{y}_t^0 - \dot{y}_t|^2\, dt - \frac{1}{2}\int_0^T |\dot{y}_t^1 - \dot{y}_t|^2\, dt\,.$$

Let $a < 0 < b$ and set

$$\tau = \inf\{T \geq 0 : F_T \notin (a,b)\} ,$$

$$\delta = \begin{cases} 1 & \text{if} \quad F_\tau = b , \\ 0 & \text{if} \quad F_\tau = a . \end{cases}$$

Theorem 5.3 *Assume that (5.2) holds. Then for any threshold policy $u = (\tau, \delta)$ with $a < 0 < b$, we have $\tau < \infty$ and*

$$\delta = 1 \qquad \text{if and only if } \mathrm{H}_1 \text{ is valid} ,$$

$$\delta = 0 \qquad \text{if and only if } \mathrm{H}_0 \text{ is valid} .$$

PROOF. Under H_1 , $y_t = y_t^1$ and for $T > 0$

$$F_T = \tfrac{1}{2} \int_0^T |\dot{y}_t^0 - \dot{y}_t^1|^2 \, dt \geq 0 .$$

By (5.2), there exists $T_1 > 0$ such that $F_{T_1} = b$. Consequently $\tau \leq T_1$ and $\delta = 1$.

Similarly, under H_0 , $y_t = y_t^0$ and for $T > 0$

$$F_T = -\tfrac{1}{2} \int_0^T |\dot{y}_t^1 - \dot{y}_t^0|^2 \, dt \leq 0.$$

We conclude again $\tau < \infty$ and $\delta = 0$. □

Thus a deterministic threshold policy always makes the correct decision under the (stronger) identifiability condition (5.2).

To compute u^ϵ (approximately), Baras and LaVigna [1] use a numerical solution of the Zakai equation. The above suggests an approximation when $\epsilon \downarrow 0$ is small. Now

$$F_T = F_T(y^0, y^1; y) .$$

Compute approximations $\tilde{y}^0, \tilde{y}^1$ to y^0, y^1 by numerically integrating the differential system (3.1). Set

$$\tilde{F}_T^\epsilon = F_T(\tilde{y}^0, \tilde{y}^1; Y) ,$$

where Y is the noisy observation record. Now define, for $a < 0 < b$

$$\tilde{\tau}^\epsilon = \inf\{T \geq 0 : \tilde{F}_T^\epsilon \notin (a,b)\} ,$$

$$\tilde{\delta}^\epsilon = \begin{cases} 1 & \text{if} \quad \tilde{F}_{\tilde{\tau}^\epsilon}^\epsilon = b , \\ 0 & \text{if} \quad \tilde{F}_{\tilde{\tau}^\epsilon}^\epsilon = a . \end{cases}$$

If the integration is sufficiently accurate, then we expect for $\alpha = 0, 1$

$$P_{\alpha,\varepsilon}(\tilde{\delta}^{\varepsilon} = \delta^{\varepsilon}) \xrightarrow{\varepsilon \downarrow 0} 0 \,.$$

Note that $|a|, b$ can be increased to increase the level of confidence.

Remark 5.4 In practice, the initial condition x_0 is not known, so that one would have to estimate x_0 also, for instance using an observer.

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A Appendix

This Appendix contains some technical results used in the paper, and a proof of Lemma 2.1.

Lemma A.1 *Let $\Lambda \subset \mathbf{R}^p$ be compact. For any $\phi \in C(\Lambda, \mathbf{R})$ define the set*

$$M(\phi) \triangleq \operatorname*{argmin}_{\lambda \in \Lambda} \phi(\lambda) .$$

Let $f, g \in C(\Lambda, \mathbf{R})$. Then for all $\alpha > 0$ there exists $\beta > 0$ such that

$$\sup_{\lambda \in \Lambda} |f(\lambda) - g(\lambda)| < \beta \quad \textit{implies} \quad \forall \lambda \in M(g) \ , \ d(\lambda, M(f)) < \alpha .$$

Proof. If not, there exists $\alpha > 0$ and a sequence $\{g_i \, , \, i \geq 0\}$ such that

$$\sup_{\lambda \in \Lambda} |f(\lambda) - g_i(\lambda)| \to 0 \quad \text{as } i \to \infty \ ,$$

and

$$d(\widehat{\lambda}_i, M(f)) \geq \alpha \quad \text{for some } \widehat{\lambda}_i \in M(g_i) \ .$$

Since Λ is compact, we can assume that $\widehat{\lambda}_i \to \lambda^* \in \Lambda$ as $i \to \infty$. Consequently

$$d(\lambda^*, M(f)) \geq \alpha \ . \tag{A.1}$$

Let $\widehat{\lambda}(f) \in M(f)$. Then

$$\begin{aligned} f(\widehat{\lambda}_i) &= f(\widehat{\lambda}(f)) + [g_i(\widehat{\lambda}(f)) - f(\widehat{\lambda}(f))] + [g_i(\widehat{\lambda}_i) - g_i(\widehat{\lambda}(f))] + [f(\widehat{\lambda}_i) - g_i(\widehat{\lambda}_i)] \\ &\leq f(\widehat{\lambda}(f)) + [g_i(\widehat{\lambda}(f)) - f(\widehat{\lambda}(f))] + [f(\widehat{\lambda}_i) - g_i(\widehat{\lambda}_i)] \\ &\leq f(\widehat{\lambda}(f)) + 2 \sup_{\lambda \in \Lambda} |f(\lambda) - g_i(\lambda)| \ , \end{aligned}$$

Sending $i \to \infty$ we obtain $f(\lambda^*) \leq f(\widehat{\lambda}(f))$. That is $\lambda^* \in M(f)$ which contradicts (A.1).
□

Lemma A.2 *Let $\Lambda \subset \mathbf{R}^p$ be compact, and $F^\lambda \in C(\mathbf{R}^m, \mathbf{R})$ be such that*

(a) there are constants $C > 0$, $C' > 0$ such that, for all $z \in \mathbf{R}^m$, $\lambda \in \Lambda$

$$F^\lambda(z) \geq C|z| - C' \ ,$$

(b) for all $R > 0$, $\delta > 0$ there exists $\gamma > 0$ such that $|\lambda' - \lambda| < \gamma$ implies

$$\sup_{z \in B(0,R)} |F^{\lambda}(z) - F^{\lambda'}(z)| < \delta \ .$$

Define $m^{\lambda} \triangleq \inf_{z \in \mathbf{R}^m} F^{\lambda}(z)$. Then

(i) there exists a constant $R > 0$ such that, for all $\lambda \in \Lambda$

$$\operatorname*{argmin}_{z \in \mathbf{R}^m} F^{\lambda}(z) \subset B(0, R) \ ,$$

(ii) the mapping $\lambda \mapsto m^{\lambda}$ is continuous.

Proof. For any $\lambda \in \Lambda$ let $z^{\lambda} \in \operatorname*{argmin}_{z \in \mathbf{R}^m} F^{\lambda}(z)$. The existence of z^{λ} follows from the continuity of F^{λ} and the coercivity hypothesis (a). Moreover

$$m^{\lambda} = F^{\lambda}(z^{\lambda}) \geq C|z^{\lambda}| - C' \ ,$$

and thus for all $\lambda \in \Lambda$

$$|z^{\lambda}| \leq \frac{m^{\lambda} + C'}{C} \ .$$

Fix $\lambda_0 \in \Lambda$. By (b) for each $\delta > 0$ there exists $\gamma > 0$ such that $|\lambda - \lambda_0| < \gamma$ implies

$$m^{\lambda} \leq F^{\lambda}(z^{\lambda_0}) = m^{\lambda_0} + [F^{\lambda}(z^{\lambda_0}) - F^{\lambda_0}(z^{\lambda_0})] \leq m^{\lambda_0} + \delta \ .$$

Then $|\lambda - \lambda_0| < \gamma$ implies

$$z^{\lambda} \in B(0, R) \ , \quad \text{with} \quad R \triangleq \frac{m^{\lambda_0} + \delta + C'}{C} \ ,$$

which proves (i).

By (b) again, this implies

$$\begin{aligned} m^{\lambda_0} \leq F^{\lambda_0}(z^{\lambda}) &= m^{\lambda} + [F^{\lambda_0}(z^{\lambda}) - F^{\lambda}(z^{\lambda})] \\ &\leq m^{\lambda} + \sup_{z \in B(0,R)} |F^{\lambda_0}(z) - F^{\lambda}(z)| \leq m^{\lambda} + \delta \ , \end{aligned}$$

and the proof of the lemma is now complete. □

The next lemma is a variant of Laplace's asymptotic method.

Lemma A.3 *Let $\Lambda \subset \mathbf{R}^p$ be compact, and $F^\lambda, G^\lambda \in C(\mathbf{R}^m, \mathbf{R})$ be such that*

(a) there are constants $C > 0$, $C' > 0$ such that, for all $z \in \mathbf{R}^m$, $\lambda \in \Lambda$

$$F^\lambda(z) \geq C|z| - C' , \quad G^\lambda(z) \geq C|z| - C' ,$$

(b) for all $R > 0$, $\delta > 0$ there exists $\gamma > 0$ such that $|\lambda' - \lambda| < \gamma$ implies

$$\sup_{z \in B(0,R)} |F^\lambda(z) - F^{\lambda'}(z)| < \delta , \quad \sup_{z \in B(0,R)} |G^\lambda(z) - G^{\lambda'}(z)| < \delta .$$

Let ρ denote a metric on $C(\mathbf{R}^m, \mathbf{R})$ corresponding to uniform convergence on compact sets.

Then, for all $\delta > 0$ there exists $\beta > 0$, $\varepsilon_0 > 0$ (depending on G) such that $0 < \varepsilon \leq \varepsilon_0$ and

$$\sup_{\lambda \in \Lambda} \rho(F^\lambda, G^\lambda) < \beta ,$$

implies

$$\sup_{\lambda \in \Lambda} \left| \varepsilon \log \int_{\mathbf{R}^m} \exp\{-\frac{1}{\varepsilon} F^\lambda(z)\}\, dz + \inf_{z \in \mathbf{R}^m} G^\lambda(z) \right| < \delta .$$

PROOF. Define

$$m^\lambda(F) \triangleq \inf_{z \in \mathbf{R}^m} F^\lambda(z) , \quad m^\lambda(G) \triangleq \inf_{z \in \mathbf{R}^m} G^\lambda(z) .$$

Lower bound: It follows from Lemma A.2 that the mappings $\lambda \mapsto m^\lambda(F)$ and $\lambda \mapsto m^\lambda(G)$ are continuous. Further, there is a constant $R > 0$ such that

$$\operatorname*{argmin}_{z \in \mathbf{R}^m} G^\lambda(z) \subset B(0, \frac{R}{2}) ,$$

for all $\lambda \in \Lambda$. Thus we can choose $0 < \beta < \delta/12$ such that $\sup_{\lambda \in \Lambda} \rho(F^\lambda, G^\lambda) < \beta$ implies

$$\sup_{\lambda \in \Lambda} |m^\lambda(F) - m^\lambda(G)| < \tfrac{1}{3}\delta ,$$

and

$$\operatorname*{argmin}_{z \in \mathbf{R}^m} F^\lambda(z) \subset B(0, R)$$

for all $\lambda \in \Lambda$. Set

$$B_\delta^\lambda \triangleq \{z \in \mathbf{R}^m \,:\, F^\lambda(z) - m^\lambda(F) < \tfrac{1}{3}\delta\} .$$

Increasing R if necessary, $B_\delta^\lambda \subset B(0, R)$ for all $\lambda \in \Lambda$ by the uniform coercivity hypothesis (a).

Now $(z,\lambda) \mapsto G^\lambda(z)$ is uniformly continuous on $B(0,R) \times \Lambda$, so there exists $r > 0$ such that

$$|z - z'| + |\lambda - \lambda'| < r \quad \text{implies} \quad |G^\lambda(z) - G^{\lambda'}(z')| < \tfrac{1}{6}\delta \ ,$$

and also, since $0 < \beta < \frac{1}{12}\delta$

$$|F^\lambda(z) - F^{\lambda'}(z')| \le 2\tfrac{1}{12}\delta + \tfrac{1}{6}\delta = \tfrac{1}{3}\delta \ ,$$

for any $z, z' \in B(0,R)$ and any $\lambda, \lambda' \in \Lambda$.

Let $z^\lambda \in \operatorname*{argmin}_{z \in \mathbf{R}^m} F^\lambda(z)$. Then $z^\lambda \in B(0,R)$ and

$$|z - z^\lambda| < r \quad \text{implies} \quad |F^\lambda(z) - m^\lambda(F)| < \tfrac{1}{3}\delta \ ,$$

for all $\lambda \in \Lambda$. That is $B(z^\lambda, r) \subset B_\delta^\lambda$ for all $\lambda \in \Lambda$. Therefore

$$\infty > v_R \ge \mu(B_\delta^\lambda) \ge v_r > 0 \ ,$$

where μ denotes the Lebesgue measure in $\mathbf{R}^m$, and v_r (resp. v_R) denotes the Lebesgue measure of a ball of radius r (resp. R) in $\mathbf{R}^m$.

Now

$$\begin{aligned} a^\lambda(\varepsilon) &\triangleq \int_{\mathbf{R}^m} \exp\{-\frac{1}{\varepsilon} F^\lambda(z)\}\, dz \\ &\ge \int_{B_\delta^\lambda} \exp\{-\frac{1}{\varepsilon} F^\lambda(z)\}\, dz \ge \mu(B_\delta^\lambda)\ \exp\{-\frac{1}{\varepsilon}(m^\lambda(F) + \tfrac{1}{3}\delta)\} \ , \end{aligned}$$

and

$$\begin{aligned} \varepsilon \log a^\lambda(\varepsilon) &\ge \varepsilon \log v_r - m^\lambda(F) - \tfrac{1}{3}\delta \\ &\ge \varepsilon \log v_r - m^\lambda(G) - \tfrac{2}{3}\delta \ge -m^\lambda(G) - \delta \ , \end{aligned}$$

provided $0 < \varepsilon \le \varepsilon_1$ for some ε_1 independent of $\lambda \in \Lambda$.

Upper bound: Let $0 < \nu < 1$. The uniform coercivity hypothesis (a) implies

$$\begin{aligned} a^\lambda(\varepsilon) &\le \int_{\mathbf{R}^m} \exp\{-\frac{1-\nu}{\varepsilon} F^\lambda(z)\}\ \exp\{-\frac{\nu}{\varepsilon} F^\lambda(z)\}\, dz \\ &\le \exp\{-\frac{1-\nu}{\varepsilon} m^\lambda(F)\} \int_{\mathbf{R}^m} \exp\{-\frac{\nu}{\varepsilon} F^\lambda(z)\}\, dz \\ &\le \exp\{-\frac{1-\nu}{\varepsilon} m^\lambda(F)\}\ \exp\{\frac{\nu C'}{\varepsilon}\} \int_{\mathbf{R}^m} \exp\{-\frac{\nu C}{\varepsilon}|z|\}\, dz \\ &\le \exp\{-\frac{1-\nu}{\varepsilon} m^\lambda(F)\}\ \exp\{\frac{\nu C''}{\varepsilon}\}\ (\frac{\varepsilon}{\nu C})^m \ , \end{aligned}$$

for all $\varepsilon > 0$. Therefore

$$\varepsilon \log a^{\lambda}(\varepsilon) \leq -(1-\nu)m^{\lambda}(F) + \nu C' + m\varepsilon(\log \varepsilon - \log \nu C)$$

$$\leq -m^{\lambda}(G) + \tfrac{1}{3}(1-\nu)\delta + \nu m^{\lambda}(G) + \nu C' + m\varepsilon(\log\varepsilon - \log \nu C) .$$

Choose ν so small that $\nu m^{\lambda}(G) + \nu C' < \frac{1}{3}\delta$. Next, choose $0 < \varepsilon_0 < \varepsilon_1$ such that $m\varepsilon(\log \varepsilon - \log \nu C) < \frac{1}{3}\delta$ for $0 < \varepsilon < \varepsilon_0$. Then we have:

$$\varepsilon \log a^{\lambda}(\varepsilon) \leq -m^{\lambda}(G) + \delta$$

provided $0 < \varepsilon \leq \varepsilon_0$. □

We turn now to the

PROOF OF LEMMA 2.1. From Sections 2 and 4 we have

$$\ell^{\varepsilon}(\theta) = -\varepsilon \log \int_{\mathbf{R}^m} q^{\theta,\varepsilon}(x,T)\, \exp\{\frac{1}{\varepsilon} h_{\theta}^{*}(x) Y_T\}\, dx \quad \text{a.s.},$$

where for a.e. $\omega \in \Omega$, $q^{\theta,\varepsilon} \in C_b^{1,2}(\mathbf{R}^m \times [0,T])$ and solves the "robust" Zakai equation

$$\frac{\partial}{\partial t} q^{\theta,\varepsilon}(x,t) - \tfrac{1}{2}\varepsilon \Delta q^{\theta,\varepsilon}(x,t) + \tilde{g}_{\theta}^{*}(x,t) D q^{\theta,\varepsilon}(x,t) + \frac{1}{\varepsilon}\tilde{V}^{\theta,\varepsilon}(x,t) q^{\theta,\varepsilon}(x,t) = 0 ,$$

$$q^{\theta,\varepsilon}(x,0) = p_0^{\theta,\varepsilon}(x) ,$$

with

$$\tilde{V}^{\theta,\varepsilon}(x,t) \triangleq \tilde{V}^{\theta}(x,t) + \tfrac{1}{2}\varepsilon Y_t^{*} \Delta h_{\theta}(x) + \varepsilon \operatorname{div} \tilde{g}_{\theta}(x,t) ,$$

$$\tilde{V}^{\theta}(x,t) \triangleq \tfrac{1}{2}|h_{\theta}(x)|^2 + b_{\theta}^{*} Y_t^{*} D h_{\theta}(x) - \tfrac{1}{2}(D h_{\theta}(x))^{*}\, Y_t\, Y_t^{*}\, D h_{\theta}(x) ,$$

$$\tilde{g}_{\theta}(x,t) \triangleq b_{\theta}(x) - Y_t^{*} D h_{\theta}(x) ;$$

see Davis [6]. Fix $\varepsilon > 0$ and $\omega \in \Omega$ such that the above holds. Now $|\tilde{g}_{\theta}(x,t)| \leq C$ and $|\tilde{V}^{\theta,\varepsilon}(x,t)| \leq C$ in $\mathbf{R}^m \times [0,T]$. Then

$$\frac{\partial}{\partial t} q^{\theta,\varepsilon}(x,t) - \tfrac{1}{2}\varepsilon \Delta q^{\theta,\varepsilon}(x,t) + \tilde{g}_{\theta}^{*}(x,t) D q^{\theta,\varepsilon}(x,t) - \frac{1}{\varepsilon} C q^{\theta,\varepsilon}(x,t) \leq 0 ,$$

and by the maximum principle, for all $(x,t) \in \mathbf{R}^m \times [0,T]$

$$0 \leq q^{\theta,\varepsilon}(x,t) \leq \exp\{\frac{CT}{\varepsilon}\}\, p_0^{\theta,\varepsilon}(x) \leq \exp\{-\frac{1}{\varepsilon}(C_2|x| - C_2' - CT)\} ,$$

i.e.

$$W^{\theta,\varepsilon}(x,t) \geq C_2|x| - C_2' - CT ,$$

where C is random and satisfies the following estimate

$$C \leq C_0 \sup_{0\leq t\leq T} |Y_t - y_t^\alpha|^2 + C_\alpha \, .$$

Therefore, by the Lebesgue dominated convergence theorem, it is enough to show that if $\theta_k \to \theta_0$ in Θ as $k \to \infty$, then $q^{\theta_k,\varepsilon}(x,T) \to q^{\theta_0,\varepsilon}(x,T)$ for each $x \in \mathbf{R}^m$. The difference $z \triangleq q^{\theta_k,\varepsilon} - q^{\theta_0,\varepsilon}$ satisfies

$$\frac{\partial}{\partial t} z(x,t) - \tfrac{1}{2}\varepsilon \Delta z(x,t) + \tilde{g}^*_{\theta_k}(x,t) Dz(x,t) + \frac{1}{\varepsilon}\tilde{V}^{\theta_k,\varepsilon}(x,t) z(x,t)$$

$$= -[\tilde{g}_{\theta_k}(x,t) - \tilde{g}_{\theta_0}(x,t)]^* Dq^{\theta_0,\varepsilon}(x,t) - \frac{1}{\varepsilon}[\tilde{V}^{\theta_k,\varepsilon}(x,t) - \tilde{V}^{\theta_0,\varepsilon}(x,t)]\, q^{\theta_0,\varepsilon}(x,t) \, ,$$

and hence

$$\frac{\partial}{\partial t} z(x,t) - \tfrac{1}{2}\varepsilon \Delta z(x,t) + \tilde{g}^*_{\theta_k}(x,t) Dz(x,t) - \frac{1}{\varepsilon} C z(x,t) \leq C_{\theta_0} \rho(\theta_k,\theta_0)(1 + \frac{1}{\varepsilon}) \, ,$$

where $\rho(\theta_k,\theta_0) \to 0$ as $k \to \infty$. Then by the maximum principle

$$z(x,t) \leq \exp\{\frac{CT}{\varepsilon}\}\, z(x,0) + T\, \exp\{\frac{CT}{\varepsilon}\}\, C_{\theta_0} \rho(\theta_k,\theta_0)(1 + \frac{1}{\varepsilon}) \, .$$

Now

$$z(x,0) \leq \frac{C}{\varepsilon}\, \exp\{-\frac{1}{\varepsilon}(C_2|x| - C_2')\}\, |S_0^{\theta_0}(x) - S_0^{\theta_k}(x)| \, .$$

Consequently, sending $k \to \infty$ we obtain

$$\limsup_{k\to\infty} \left\{ q^{\theta_k,\varepsilon}(x,T) - q^{\theta_0,\varepsilon}(x,T) \right\} \leq 0 \, .$$

Similarly, we obtain the reverse inequality and conclude. □

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